

# SINGULARITY CATEGORIES OF DERIVED CATEGORIES OF HEREDITARY ALGEBRAS ARE DERIVED CATEGORIES

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**ABSTRACT.** We show that for the path algebra  $A$  of an acyclic quiver, the singularity category of the derived category  $D^b(\text{mod } A)$  is triangle equivalent to the derived category of the functor category of  $\underline{\text{mod}} A$ , that is,  $D_{\text{sg}}(D^b(\text{mod } A)) \simeq D^b(\text{mod}(\underline{\text{mod}} A))$ . This extends a result in [IO] for the path algebra  $A$  of a Dynkin quiver. An important step is to establish a functor category analog of Happel's triangle equivalence for repetitive algebras.

## 1. INTRODUCTION

Let  $k$  be a field and  $A$  be a finite dimensional  $k$ -algebra. In [IO], it was shown that if  $A$  is a representation finite hereditary algebra, then there exists a triangle equivalence

$$\underline{\text{mod}} D^b(\text{mod } A) \simeq D^b(\text{mod } B), \quad (1.1)$$

where  $B$  is the stable Auslander algebra of  $A$ ,  $\text{mod } D^b(\text{mod } A)$  is the Frobenius category of finitely presented functors from  $D^b(\text{mod } A)$  to  $\mathcal{A}b$ , and  $\underline{\text{mod}} D^b(\text{mod } A)$  is its stable category.

In this paper, we extend a triangle equivalence (1.1) to the case when  $A$  is a representation infinite hereditary algebra. In this case, the role of the stable Auslander algebra is played by the category  $\text{mod}(\underline{\text{mod}} A)$  of finitely presented functors from the stable category  $\underline{\text{mod}} A$  to  $\mathcal{A}b$ . Our main result is the following.

**Theorem 1.1** (Theorem 4.5). *Let  $A$  be a hereditary algebra. We have a triangle equivalence*

$$\underline{\text{mod}} D^b(\text{mod } A) \simeq D^b(\text{mod}(\underline{\text{mod}} A)). \quad (1.2)$$

Note that for a triangulated category  $\mathcal{T}$ , the stable category  $\underline{\text{mod}} \mathcal{T}$  is triangle equivalent to the singularity category  $D_{\text{sg}}(\mathcal{T}) = D^b(\text{mod } \mathcal{T})/K^b(\text{proj } \mathcal{T})$  [Bu, O] (see Theorem 2.17). Thus (1.2) can be rewritten as  $D_{\text{sg}}(D^b(\text{mod } A)) \simeq D^b(\text{mod}(\underline{\text{mod}} A))$ .

To prove Theorem 1.1, we need to give general preliminary results on functor categories and repetitive categories. The functor category  $\text{mod}(\underline{\text{mod}} A)$  is an abelian category with enough projectives and enough injectives since the category  $\underline{\text{mod}} A$  forms a dualizing  $k$ -variety, which is a distinguished class of  $k$ -linear categories introduced by Auslander and Reiten [AR74]. A key role is played by the repetitive category  $R(\underline{\text{mod}} A)$  of  $\underline{\text{mod}} A$ . The following our first result implies that  $R(\underline{\text{mod}} A)$  is a dualizing  $k$ -variety.

**Theorem 1.2** (Theorem 3.7). *Let  $\mathcal{A}$  be a dualizing  $k$ -variety. Then  $R\mathcal{A}$  is a dualizing  $k$ -variety.*

In particular,  $\text{mod } R\mathcal{A}$  is a Frobenius abelian category for any dualizing  $k$ -variety  $\mathcal{A}$ . We denote by  $\underline{\text{mod}} R\mathcal{A}$  the stable category of  $\text{mod } R\mathcal{A}$ , which is triangulated.

In the case where  $A$  is a representation finite hereditary algebra, the following Happel's theorem [H] played an important role in the proof of a triangle equivalence (1.1): for a finite dimensional  $k$ -algebra  $A$  of finite global dimension, the bounded derived category of  $A$  is triangle equivalent to the stable category of the repetitive algebra of  $A$ . In Section 3, we show a categorical analog of this triangle equivalence for dualizing  $k$ -varieties. In fact, we deal with the following more general class of categories including dualizing  $k$ -varieties. For a  $k$ -linear additive category  $\mathcal{A}$ , we denote by  $\text{proj } \mathcal{A}$  the category of finitely generated projective  $\mathcal{A}$ -modules and by  $\text{mod } \mathcal{A}$  the category of  $\mathcal{A}$ -modules having resolutions by  $\text{proj } \mathcal{A}$ . We consider the following conditions:

(IFP)  $D\mathcal{A}(X, -)$  is in  $\text{mod } \mathcal{A}$  for each  $X \in \mathcal{A}$ , where  $D = \text{Hom}_k(-, k)$ .

(G)  $D\mathcal{A}(X, -)$  has finite projective dimension over  $\mathcal{A}$  for each  $X \in \mathcal{A}$ .

For example, if  $\mathcal{A}$  is a dualizing  $k$ -variety, then  $\mathcal{A}$  satisfies the condition (IFP). On the other hand, the condition (G) is a categorical version of Gorensteinness. Gorenstein-projective modules (also known as Cohen-Macaulay modules, totally reflexive modules) are important class of modules. We denote by  $\text{GP}(\text{RA}, \mathcal{A})$  the category of Gorenstein-projective  $\text{RA}$ -modules of finite projective dimension as  $\mathcal{A}$ -modules. We prove the following.

**Theorem 1.3** (Corollaries 3.17, 3.18). *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category.*

(a) *Assume that  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (IFP) and (G). Then we have a triangle equivalence*

$$K^b(\text{proj } \mathcal{A}) \simeq \underline{\text{GP}}(\text{RA}, \mathcal{A}).$$

(b) *Assume that  $\mathcal{A}$  is a dualizing  $k$ -variety. If each object of  $\text{mod } \mathcal{A}$  and  $\text{mod } \mathcal{A}^{\text{op}}$  has finite projective dimension, then we have a triangle equivalence*

$$D^b(\text{mod } \mathcal{A}) \simeq \underline{\text{mod}} \text{RA}.$$

We refer to [BGG, IO, K1, K2, Lu, MY, MU, Y] for recent results which realize stable categories as derived categories in different settings.

In Section 4, we show the following theorem, which together with Theorem 1.3 implies Theorem 1.1.

**Theorem 1.4** (Theorem 4.3). *Let  $A$  be a representation infinite hereditary algebra. Then we have an equivalence of additive categories*

$$R(\underline{\text{mod}} A) \simeq D^b(\text{mod } A).$$

**Notation.** In this paper, we denote by  $k$  a field. All subcategories are full and closed under isomorphisms. Let  $\mathcal{C}$  be an additive category and  $\mathcal{S}$  be a subclass of objects of  $\mathcal{C}$  or a subcategory of  $\mathcal{C}$ . We denote by  $\text{add } \mathcal{S}$  the subcategory of  $\mathcal{C}$  whose objects are direct summands of finite direct sums of objects in  $\mathcal{S}$ . For subcategories  $\mathcal{C}_i$  ( $i \in I$ ) of  $\mathcal{C}$ , we denote by  $\bigvee_{i \in I} \mathcal{C}_i$  the smallest additive subcategory of  $\mathcal{C}$  containing all  $\mathcal{C}_i$  and closed under direct summands. For objects  $X, Y \in \mathcal{C}$ , we denote by  $\mathcal{C}(X, Y)$  the set of morphisms from  $X$  to  $Y$  in  $\mathcal{C}$ . We call a category *skeletally small* if the class of isomorphism class of objects is a set. We assume that all categories in this paper are skeletally small.

## 2. PRELIMINARIES

**2.1. Functor categories.** In this subsection, we recall the definition of modules over categories. Let  $\mathcal{A}$  be an additive category. An  $\mathcal{A}$ -module is a contravariant additive functor from  $\mathcal{A}$  to  $\mathcal{A}b$ , where  $\mathcal{A}b$  is the category of abelian groups. We denote by  $\text{Mod } \mathcal{A}$  the category of  $\mathcal{A}$ -modules, where morphisms of  $\text{Mod } \mathcal{A}$  are morphisms of functors. Since  $\mathcal{A}$  is skeletally small,  $\text{Mod } \mathcal{A}$  is a category. It is well known that  $\text{Mod } \mathcal{A}$  is abelian.

For two morphisms  $f : L \rightarrow M$  and  $g : M \rightarrow N$  of  $\text{Mod}\mathcal{A}$ , the sequence  $L \rightarrow M \rightarrow N$  is exact in  $\text{Mod}\mathcal{A}$  if and only if the induced sequence  $L(X) \rightarrow M(X) \rightarrow N(X)$  is exact in  $\mathcal{A}b$  for any  $X \in \mathcal{A}$ .

**Example 2.1.** For each  $X \in \mathcal{A}$ , we have an  $\mathcal{A}$ -module  $\mathcal{A}(-, X)$ . By Yoneda's lemma,  $\mathcal{A}(-, X)$  is projective in  $\text{Mod}\mathcal{A}$ .

The following notation is basic and used throughout this paper. We call an  $\mathcal{A}$ -module  $M$  *finitely generated* if there exists an epimorphism  $\mathcal{A}(-, X) \rightarrow M$  in  $\text{Mod}\mathcal{A}$  for some  $X \in \mathcal{A}$ . We denote by  $\text{proj}\mathcal{A}$  the subcategory of  $\text{Mod}\mathcal{A}$  consisting of all finitely generated projective  $\mathcal{A}$ -modules. Note that finitely generated projective modules are precisely direct summands of representable functors. We need the following notation which is called  $FP_n$  in some literatures (e.g. [BGI, Br]).

**Definition 2.2.** Let  $\mathcal{A}$  be an additive category and  $n \geq 0$  be an integer.

- (1) We denote by  $\text{mod}_n\mathcal{A}$  the subcategory of  $\text{Mod}\mathcal{A}$  consisting of all  $\mathcal{A}$ -modules  $M$  such that there exists an exact sequence

$$P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

in  $\text{Mod}\mathcal{A}$ , where  $P_i$  is in  $\text{proj}\mathcal{A}$  for each  $0 \leq i \leq n$ .

- (2) We denote by  $\text{mod}\mathcal{A}$  the subcategory of  $\text{Mod}\mathcal{A}$  consisting of all  $\mathcal{A}$ -modules  $M$  such that there exists an exact sequence

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

in  $\text{Mod}\mathcal{A}$ , where  $P_i$  is in  $\text{proj}\mathcal{A}$  for each  $i \geq 0$ .

The following lemma is a basic observation on  $\text{mod}_n\mathcal{A}$ .

**Lemma 2.3.** *The following statements hold for an additive category  $\mathcal{A}$ .*

- (a) *Let  $M \in \text{mod}_n\mathcal{A}$ . Assume that there exists an exact sequence  $P_l \rightarrow P_{l-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$  with  $P_i \in \text{proj}\mathcal{A}$  and  $l \leq n$ . Then there exist  $P_{l+1}, \dots, P_n \in \text{proj}\mathcal{A}$  and an exact sequence  $P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$ .*
- (b) *Let  $M \in \text{Mod}\mathcal{A}$ . Assume that there exist the following two exact sequences*

$$0 \rightarrow K \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0,$$

$$0 \rightarrow L \rightarrow Q_n \rightarrow Q_{n-1} \rightarrow \cdots \rightarrow Q_0 \rightarrow M \rightarrow 0,$$

*where  $P_i, Q_i \in \text{proj}\mathcal{A}$  for each  $i \geq 0$ . Then there exist  $P, Q \in \text{proj}\mathcal{A}$  such that  $K \oplus P \simeq L \oplus Q$ .*

*Proof.* (a) This follows from (b).

(b) The case where  $n = 0$  is well known as Schanuel's Lemma. The case where  $n > 0$  is shown by an induction on  $n$  and by using the case where  $n = 0$ .  $\square$

The following lemma gives a sufficient condition when an  $\mathcal{A}$ -module is in  $\text{mod}_n\mathcal{A}$ . For simplicity, we use the notation  $\text{mod}_{-1}\mathcal{A} := \text{Mod}\mathcal{A}$ ,  $\text{mod}_\infty\mathcal{A} := \text{mod}\mathcal{A}$  and  $\infty - 1 := \infty$ .

**Lemma 2.4.** *Let  $\mathcal{A}$  be an additive category and  $M$  be an  $\mathcal{A}$ -module. Then we have the following properties.*

- (a) *Let  $n \geq 0$  be an integer. If there exists an exact sequence  $X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 \rightarrow M \rightarrow 0$  in  $\text{Mod}\mathcal{A}$  with  $X_i \in \text{mod}_{n-i}\mathcal{A}$  for any  $0 \leq i \leq n$ , then we have  $M \in \text{mod}_n\mathcal{A}$ .*
- (b) *If there exists an exact sequence  $\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$  in  $\text{Mod}\mathcal{A}$  with  $X_i \in \text{mod}\mathcal{A}$  for any  $i \geq 0$ , then we have  $M \in \text{mod}\mathcal{A}$ .*

- (c) Let  $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . For an exact sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  in  $\text{Mod}\mathcal{A}$  with  $L \in \text{mod}_{n-1}\mathcal{A}$  and  $M \in \text{mod}_n\mathcal{A}$ , we have  $N \in \text{mod}_n\mathcal{A}$ .

*Proof.* (a) We have the following commutative diagram

$$\begin{array}{ccccccc}
 X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 \longrightarrow M \longrightarrow 0 \\
 \uparrow & & \uparrow & & & & \uparrow \\
 P_{n,0} & \longrightarrow & P_{n-1,0} & \longrightarrow & \cdots & \longrightarrow & P_{0,0} \\
 & & \uparrow & & & & \uparrow \\
 & & P_{n-1,1} & \longrightarrow & \cdots & \longrightarrow & P_{0,1} \\
 & & & & & & \uparrow \\
 & & & & & & \cdots \\
 & & & & & & \uparrow \\
 & & & & & & P_{0,n}
 \end{array}$$

in  $\text{Mod}\mathcal{A}$ , where each  $P_{i,0} \rightarrow X_i$  is epimorphism for  $0 \leq i \leq n$ , each vertical sequence is exact and each  $P_{i,j}$  is in  $\text{proj}\mathcal{A}$ . Thus we have an exact sequence

$$\overline{P}_n \rightarrow \cdots \rightarrow \overline{P}_1 \rightarrow \overline{P}_0 \rightarrow M \rightarrow 0$$

in  $\text{Mod}\mathcal{A}$ , where  $\overline{P}_i = \bigoplus_{j=0}^i P_{j,i-j}$  for  $0 \leq i \leq n$ . Since  $\overline{P}_i$  is in  $\text{proj}\mathcal{A}$  for  $0 \leq i \leq n$ ,  $M$  is an object of  $\text{mod}_n\mathcal{A}$ .

(b) This comes from the same argument as (a).

(c) This follows from (a) for  $n \in \mathbb{Z}_{\geq 0}$  and (b) for  $n = \infty$ .  $\square$

Let  $\mathcal{A}$  be an abelian category and  $\mathcal{B}$  be a subcategory of  $\mathcal{A}$ . We say that  $\mathcal{B}$  is a *thick* subcategory of  $\mathcal{A}$  if  $\mathcal{B}$  is closed under direct summands and for any exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\mathcal{A}$ , if two of  $X, Y, Z$  are in  $\mathcal{A}$ , then so is the third. We have the following observation of the categories  $\text{mod}_n\mathcal{A}$ .

**Lemma 2.5.** *Let  $\mathcal{A}$  be an additive category. Then we have the following statements.*

- (a)  $\text{mod}_n\mathcal{A}$  is closed under extensions and direct summands in  $\text{Mod}\mathcal{A}$  for each  $n \geq 0$ .
- (b)  $\text{mod}\mathcal{A} = \bigcap_{n \geq 0} \text{mod}_n\mathcal{A}$  holds.
- (c) (e.g. [E, Proposition 2.6])  $\text{mod}\mathcal{A}$  is a thick subcategory of  $\text{Mod}\mathcal{A}$ .

*Proof.* (a) By Horseshoe Lemma,  $\text{mod}_n\mathcal{A}$  is closed under extensions in  $\text{Mod}\mathcal{A}$ . Let  $X \oplus Y \in \text{mod}_n\mathcal{A}$ . We show that  $X, Y \in \text{mod}_n\mathcal{A}$  by an induction on  $n$ . If  $n = 0$ , then the claim is clear. Assume  $n > 0$ . Since  $X \oplus Y \in \text{mod}_n\mathcal{A} \subset \text{mod}_{n-1}\mathcal{A}$  holds, by the inductive hypothesis, we have  $X, Y \in \text{mod}_{n-1}\mathcal{A}$ . Then by Lemma 2.4 (c), we have  $X, Y \in \text{mod}_n\mathcal{A}$ .

(b) In general  $\text{mod}\mathcal{A} \subset \text{mod}_n\mathcal{A}$  holds for each  $n \geq 0$ . The converse follows from Lemma 2.3 (a).

(c) By (a) and (b),  $\text{mod}\mathcal{A}$  is closed under extensions and direct summands. Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be an exact sequence in  $\text{Mod}\mathcal{A}$ . By Lemma 2.4 (c), if  $L, M \in \text{mod}\mathcal{A}$ , then  $N \in \text{mod}\mathcal{A}$  holds. Assume that  $M, N \in \text{mod}\mathcal{A}$ . There exists an exact sequence  $0 \rightarrow \Omega N \rightarrow P \rightarrow N \rightarrow 0$  such that  $P \in \text{proj}\mathcal{A}$  and  $\Omega N \in \text{mod}\mathcal{A}$ . By taking a pull-back diagram of  $M \rightarrow N \leftarrow P$ , we have an exact sequence  $0 \rightarrow \Omega N \rightarrow P \oplus L \rightarrow M \rightarrow 0$ . Since  $\text{mod}\mathcal{A}$  is closed under extensions and direct summands, we have  $L \in \text{mod}\mathcal{A}$ .  $\square$

**2.2. Gorenstein-projective modules.** We define Gorenstein-projective modules. Let  $\mathcal{A}$  be an additive category. We first define a contravariant functor

$$(-)^* : \text{Mod}\mathcal{A} \rightarrow \text{Mod}\mathcal{A}^{\text{op}}$$

as follows: for  $M \in \text{Mod } \mathcal{A}$  and  $X \in \mathcal{A}$ , let  $(M)^*(X) := (\text{Mod } \mathcal{A})(M, \mathcal{A}(-, X))$ . By the same way, we define a contravariant functor  $(-)^* : \text{Mod } \mathcal{A}^{\text{op}} \rightarrow \text{Mod } \mathcal{A}$ . Let  $P_\bullet := (P_i, d_i : P_i \rightarrow P_{i+1})_{i \in \mathbb{Z}}$  be a complex of finitely generated projective  $\mathcal{A}$ -modules. We say that  $P_\bullet$  is *totally acyclic* if complexes  $P_\bullet$  and  $\cdots \rightarrow (P_{i+1})^* \rightarrow (P_i)^* \rightarrow (P_{i-1})^* \rightarrow \cdots$  are acyclic.

**Definition 2.6.** Let  $\mathcal{A}$  be an additive category. An  $\mathcal{A}$ -module  $M$  is said to be *Gorenstein-projective* if there exists a totally acyclic complex  $P_\bullet$  such that  $\text{Im } d_0$  is isomorphic to  $M$ . We denote by  $\text{GPA}$  the full subcategory of  $\text{Mod } \mathcal{A}$  consisting of all Gorenstein-projective  $\mathcal{A}$ -modules.

For instance, a finitely generated projective  $\mathcal{A}$ -module is Gorenstein-projective. In general,  $\text{GPA} \subset \text{mod } \mathcal{A}$  holds. We see a fundamental properties of Gorenstein-projective modules.

Let  $\mathcal{W}$  be a subcategory of  $\text{Mod } \mathcal{A}$ . We denote by  ${}^\perp \mathcal{W}$  the subcategory of  $\text{Mod } \mathcal{A}$  consisting of  $\mathcal{A}$ -modules  $M$  satisfying  $\text{Ext}_{\text{Mod } \mathcal{A}}^i(M, W) = 0$  for any  $W \in \mathcal{W}$  and any  $i > 0$ . We denote by  $\mathcal{X}_{\mathcal{W}}$  the subcategory of  ${}^\perp \mathcal{W}$  consisting of  $\mathcal{A}$ -modules  $M$  such that there exists an exact sequence  $0 \rightarrow M \rightarrow W_0 \xrightarrow{f_0} W_1 \xrightarrow{f_1} \cdots$  with  $W_i \in \mathcal{W}$  and  $\text{Im } f_i \in {}^\perp \mathcal{W}$  for any  $i \geq 0$ . By [AR91, Proposition 5.1],  $\mathcal{X}_{\text{proj } \mathcal{A}}$  is closed under extensions, direct summands and kernels of epimorphisms in  $\text{Mod } \mathcal{A}$ .

**Lemma 2.7.** *Let  $\mathcal{A}$  be an additive category. Then the following holds.*

- (a) *The functor  $(-)^* : \text{Mod } \mathcal{A} \rightarrow \text{Mod } \mathcal{A}^{\text{op}}$  induces a duality  $(-)^* : \text{GPA} \rightarrow \text{GPA}^{\text{op}}$ .*
- (b)  *$\mathcal{X}_{\text{proj } \mathcal{A}} \cap \text{mod } \mathcal{A} = \text{GPA}$  holds. In particular,  $\text{GPA}$  is closed under extensions, direct summands and kernels of epimorphisms in  $\text{Mod } \mathcal{A}$ .*

*Proof.* (a) This follows from the definition of  $\text{GPA}$  and the fact that  $(-)^*$  induces a duality between  $\text{proj } \mathcal{A}$  and  $\text{proj } \mathcal{A}^{\text{op}}$ .

(b) In general  $\mathcal{X}_{\text{proj } \mathcal{A}} \cap \text{mod } \mathcal{A} \supset \text{GPA}$  holds. If  $M \in \mathcal{X}_{\text{proj } \mathcal{A}} \cap \text{mod } \mathcal{A}$ , then there exists an exact sequence  $P_\bullet = (P_i, d_i : P_i \rightarrow P_{i+1})_{i \in \mathbb{Z}}$ , where  $M \simeq \text{Im } d_0$ ,  $P_i \in \text{proj } \mathcal{A}$  for any  $i \in \mathbb{Z}$  and  $\text{Im } d_i \in {}^\perp(\text{proj } \mathcal{A})$  for any  $i \geq 1$ . Then this sequence is totally acyclic, since  $\text{Im } d_i \in {}^\perp(\text{proj } \mathcal{A})$  holds for any  $i \geq 1$ .  $\square$

Let  $\mathcal{B}$  be an extension closed subcategory of an abelian category  $\mathcal{A}$ . An exact sequence in  $\mathcal{A}$  is called an exact sequence in  $\mathcal{B}$  if each term of it is an object of  $\mathcal{B}$ . We say that an object  $Z$  in  $\mathcal{B}$  is *relative-projective* if any exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\mathcal{B}$  splits. Dually, we define *relative-injective* objects. We say that  $\mathcal{B}$  has *enough projectives* if for any  $X \in \mathcal{B}$ , there exists an exact sequence  $0 \rightarrow Z \rightarrow P \rightarrow X \rightarrow 0$  in  $\mathcal{B}$  such that  $P$  is relative-projective. Dually, we define a subcategory of  $\mathcal{A}$  which has *enough injectives*. An extension closed subcategory  $\mathcal{B}$  of  $\mathcal{A}$  is said to be *Frobenius* if  $\mathcal{B}$  has enough projectives, enough injectives and the relative-projective objects coincide with the relative-injective objects.

The following observation is immediate (cf. [C]).

**Proposition 2.8.** *Let  $\mathcal{A}$  be an additive category. Then  $\text{GPA}$  is a Frobenius category, where the relative-projective objects are precisely finitely generated  $\mathcal{A}$ -modules.*

*Proof.*  $\text{GPA}$  is extension closed in  $\text{Mod } \mathcal{A}$  by Lemma 2.7 (b). By the definition of  $\text{GPA}$  and the duality  $(-)^* : \text{GPA} \rightarrow \text{GPA}^{\text{op}}$ ,  $\text{GPA}$  has enough projectives and enough injectives. Again by the definition of  $\text{GPA}$ , the relative-projective objects coincide with the relative-injective objects, which coincide with finitely generated projective  $\mathcal{A}$ -modules.  $\square$

**2.3. Dualizing  $k$ -varieties and Serre dualities.** In this subsection, we recall the definition of dualizing  $k$ -varieties. Let  $\mathcal{A}$  be an additive category. We call an object of  $\text{mod}_1\mathcal{A}$  a *finitely presented  $\mathcal{A}$ -module*.

A morphism  $X \rightarrow Y$  in  $\mathcal{A}$  is a *weak kernel* of a morphism  $Y \rightarrow Z$  if the induced sequence  $\mathcal{A}(-, X) \rightarrow \mathcal{A}(-, Y) \rightarrow \mathcal{A}(-, Z)$  is exact in  $\text{Mod}\mathcal{A}$ . We say that  $\mathcal{A}$  has weak kernels if each morphism in  $\mathcal{A}$  has a weak kernel. The following lemma says when an additive category has weak kernels.

**Lemma 2.9.** *Let  $\mathcal{A}$  be an additive category. The following statements are equivalent.*

- (i)  $\mathcal{A}$  has weak kernels.
- (ii)  $\text{mod}_1\mathcal{A}$  is abelian.
- (iii)  $\text{mod}_1\mathcal{A} = \text{mod}\mathcal{A}$  holds.

*Proof.* It is well known that the statements (i) and (ii) are equivalent. The statements (i) and (iii) are equivalent by [E, Proposition 2.7].  $\square$

Let  $\mathcal{A}$  be an additive category and  $X \in \mathcal{A}$ . A morphism  $e : X \rightarrow X$  in  $\mathcal{A}$  is called an *idempotent* if  $e^2 = e$ . We call  $\mathcal{A}$  *idempotent complete* if each idempotent of  $\mathcal{A}$  has a kernel.

Let  $k$  be a field. A  $k$ -linear category  $\mathcal{A}$  is a category such that  $\mathcal{A}(X, Y)$  admits a structure of  $k$ -modules and the composition of morphisms of  $\mathcal{A}$  is  $k$ -bilinear. A contravariant functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  between  $k$ -linear categories are called  $k$ -functor if  $F_{X,Y} : \mathcal{A}(X, Y) \rightarrow \mathcal{B}(FY, FX)$  is  $k$ -linear for any  $X, Y \in \mathcal{A}$ . If  $\mathcal{A}$  is an additive  $k$ -linear category, then any  $\mathcal{A}$ -module can be regarded as a contravariant additive  $k$ -functor from  $\mathcal{A}$  to  $\text{Mod}k$ , where  $\text{Mod}k$  is the category of  $k$ -modules.

Let  $\mathcal{A}$  be a  $k$ -linear additive category. We call  $\mathcal{A}$  *Hom-finite* if  $\mathcal{A}(X, Y)$  is finitely generated over  $k$  for any  $X, Y \in \mathcal{A}$ . We recall one proposition about the Krull-Schmidt property of  $k$ -linear additive categories.

**Proposition 2.10.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category. Then the following properties are equivalent.*

- (i)  $\mathcal{A}$  is idempotent complete.
- (ii) The endomorphism algebra of each indecomposable object in  $\mathcal{A}$  is local.
- (iii)  $\mathcal{A}$  is Krull-Schmidt, that is, each object of  $\mathcal{A}$  is a finite direct sum of objects whose endomorphism algebras are local.

Moreover the decomposition of (iii) is unique up to isomorphism.

**Proposition 2.11.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category. Then  $\text{mod}\mathcal{A}$  is Krull-Schmidt. In particular, each object of  $\text{mod}\mathcal{A}$  has a minimal projective resolution.*

*Proof.* Since  $\text{mod}\mathcal{A}$  is closed under direct summands in  $\text{Mod}\mathcal{A}$ ,  $\text{mod}\mathcal{A}$  is idempotent complete.  $\text{mod}\mathcal{A}$  is Hom-finite, since  $\mathcal{A}$  is Hom-finite.  $\square$

We recall the definition of dualizing  $k$ -varieties. Let  $\mathcal{A}$  be a  $k$ -linear additive category. We have contravariant functors  $D : \text{Mod}\mathcal{A} \rightarrow \text{Mod}\mathcal{A}^{\text{op}}$  and  $D : \text{Mod}\mathcal{A}^{\text{op}} \rightarrow \text{Mod}\mathcal{A}$  given by  $(DM)(X) := D(M(X))$ .

**Definition 2.12.** Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite, idempotent complete additive category. We call  $\mathcal{A}$  a *dualizing  $k$ -variety* if the functor  $D : \text{Mod}\mathcal{A} \rightarrow \text{Mod}\mathcal{A}^{\text{op}}$  induces a duality between  $\text{mod}_1\mathcal{A}$  and  $\text{mod}_1\mathcal{A}^{\text{op}}$ .

The following is typical examples of dualizing  $k$ -varieties.

**Example 2.13.** [AR74]

- (a) If  $\mathcal{A}$  is a dualizing  $k$ -variety, then  $\mathcal{A}^{\text{op}}$  is a dualizing  $k$ -variety.
- (b) Let  $A$  be a finite dimensional  $k$ -algebra and  $\text{mod}A$  be the category of finitely generated  $A$ -modules. Let  $\text{proj}A$  be the full subcategory of  $\text{mod}A$  consisting of all finitely generated projective  $A$ -modules. Then  $\text{mod}A$  and  $\text{proj}A$  are dualizing  $k$ -varieties.

We state some properties of dualizing  $k$ -varieties.

**Lemma 2.14.** [AR74] *Let  $\mathcal{A}$  be a dualizing  $k$ -variety, then we have the following properties.*

- (a)  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  have weak kernels.
- (b)  $\text{mod}\mathcal{A}$  is a dualizing  $k$ -variety.
- (c) Each object in  $\text{mod}\mathcal{A}$  has a projective cover and an injective hull.

Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category. A Serre functor on  $\mathcal{A}$  is an auto-equivalence  $\mathbb{S} : \mathcal{A} \rightarrow \mathcal{A}$  such that there exists a bifunctorial isomorphism

$$\text{Hom}_{\mathcal{A}}(X, Y) \simeq D \text{Hom}_{\mathcal{A}}(Y, \mathbb{S}(X))$$

for any  $X, Y \in \mathcal{A}$ . We denote by  $\mathbb{S}^{-1}$  a quasi-inverse of  $\mathbb{S}$ . It is easy to see that if  $\mathcal{A}$  has a Serre functor  $\mathbb{S}$ , then  $\mathcal{A}^{\text{op}}$  has a Serre functor  $\mathbb{S}^{-1}$ .

If  $\mathcal{A}$  has a Serre functor  $\mathbb{S}$ , then  $(-)^*$  is described as in the following lemma. Since  $\mathbb{S}$  is an auto-equivalence, we have an equivalence  $\text{Mod}\mathcal{A} \rightarrow \text{Mod}\mathcal{A}$  given by  $M \mapsto M \circ \mathbb{S}^{-1}$ . By composing the functor  $D : \text{Mod}\mathcal{A} \rightarrow \text{Mod}\mathcal{A}^{\text{op}}$ , we have a contravariant functor  $\text{Mod}\mathcal{A} \rightarrow \text{Mod}\mathcal{A}^{\text{op}}$  given by  $M \mapsto D(M \circ \mathbb{S}^{-1})$ . We denote by  $\text{Mod}_{\text{fg}}\mathcal{A}$  the subcategory of  $\text{Mod}\mathcal{A}$  consisting of  $\mathcal{A}$ -modules  $M$  such that  $M(X)$  is finitely generated over  $k$  for any  $X \in \mathcal{A}$ . Note that  $D$  induces a duality  $\text{Mod}_{\text{fg}}\mathcal{A} \rightarrow \text{Mod}_{\text{fg}}\mathcal{A}^{\text{op}}$  and the categories  $\text{mod}_0\mathcal{A}$  and  $\text{GP}\mathcal{A}$  are contained in  $\text{Mod}_{\text{fg}}\mathcal{A}$ .

**Lemma 2.15.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category with a Serre functor  $\mathbb{S}$ . Then the following statements hold.*

- (a) *We have an isomorphism of functors  $(-)^* \simeq D(- \circ \mathbb{S}^{-1}) : \text{Mod}_{\text{fg}}\mathcal{A} \rightarrow \text{Mod}_{\text{fg}}\mathcal{A}^{\text{op}}$ , and this functor is a duality.*
- (b) *Let  $M \in \text{Mod}\mathcal{A}$ . The following statements are equivalent.*
  - (i)  $M \in \text{GP}\mathcal{A}$ .
  - (ii)  $M \in \text{mod}\mathcal{A}$  and  $M^* \in \text{mod}\mathcal{A}^{\text{op}}$ .

*Proof.* (a) Let  $M \in \text{Mod}_{\text{fg}}\mathcal{A}$  and  $X \in \mathcal{A}$ . We have the following equalities.

$$\begin{aligned} (M)^*(X) &= (\text{Mod}\mathcal{A})(M, \mathcal{A}(-, X)) \\ &\simeq (\text{Mod}\mathcal{A}^{\text{op}})(D\mathcal{A}(-, X), DM) \\ &\simeq (\text{Mod}\mathcal{A}^{\text{op}})(\mathcal{A}(\mathbb{S}^{-1}(X), -), DM) \\ &\simeq D(M(\mathbb{S}^{-1}(X))), \end{aligned}$$

which functorial on  $X$ . Thus we have an isomorphism of functors  $(-)^* \simeq D(- \circ \mathbb{S}^{-1})$ . This functor is a duality, since  $D$  is a duality and  $\mathbb{S}$  is an equivalence.

(b) Assume that  $M \in \text{GP}\mathcal{A}$ . By Lemma 2.7 (a), we have  $M^* \in \text{GP}\mathcal{A}^{\text{op}}$ . In general  $\text{GP}\mathcal{A} \subset \text{mod}\mathcal{A}$  holds, thus (i) implies (ii). Assume that (ii) holds. There exists an exact sequence  $\cdots \rightarrow Q_2 \rightarrow Q_1 \rightarrow M^* \rightarrow 0$ , where  $Q_i \in \text{proj}\mathcal{A}^{\text{op}}$ . By (a),  $(-)^*$  is an exact functor. Therefore we have an exact sequence

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{d} Q_1^* \rightarrow Q_2^* \rightarrow \cdots,$$

where  $P_i, Q_i^* \in \text{proj } \mathcal{A}$  and  $\text{Im } d \simeq M$ . This exact sequence is totally acyclic, since  $(-)^*$  is exact. We have  $M \in \text{GP}\mathcal{A}$ .  $\square$

Later we use the following characterization of dualizing  $k$ -varieties with Serre functors.

**Proposition 2.16.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite, idempotent complete additive category. Then the following statements are equivalent.*

- (i)  $\mathcal{A}$  is a dualizing  $k$ -variety and has a Serre functor.
- (ii)  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  have weak kernels and  $\mathcal{A}$  has a Serre functor.
- (iii)  $\text{GP}\mathcal{A} = \text{mod}_1 \mathcal{A}$ ,  $\text{GP}\mathcal{A}^{\text{op}} = \text{mod}_1 \mathcal{A}^{\text{op}}$  hold and  $\text{DA}(X, -) \in \text{mod}_1 \mathcal{A}$ ,  $\text{DA}(-, X) \in \text{mod}_1 \mathcal{A}^{\text{op}}$  hold for any  $X \in \mathcal{A}$ .

*Proof.* By Lemma 2.14, (i) implies (ii). We show that (ii) implies (i). Let  $M \in \text{mod}_1 \mathcal{A}$ . We show that  $DM$  is in  $\text{mod}_1 \mathcal{A}^{\text{op}}$ . There exists an exact sequence  $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$  for some  $P_1, P_0 \in \text{proj } \mathcal{A}$ . By the functor  $D : \text{Mod } \mathcal{A} \rightarrow \text{Mod } \mathcal{A}^{\text{op}}$ , we have an exact sequence  $0 \rightarrow DM \rightarrow DP_0 \rightarrow DP_1$  in  $\text{Mod } \mathcal{A}$ . Since  $\mathcal{A}$  has a Serre functor, we have  $DP_1, DP_0 \in \text{proj } \mathcal{A}^{\text{op}}$ . Since  $\mathcal{A}^{\text{op}}$  has weak kernels,  $DM$  is in  $\text{mod}_1 \mathcal{A}^{\text{op}}$ . By the dual argument, for any  $N \in \text{mod}_1 \mathcal{A}^{\text{op}}$ , we have  $DN \in \text{mod}_1 \mathcal{A}$ . Thus  $D : \text{mod}_1 \mathcal{A} \rightarrow \text{mod}_1 \mathcal{A}^{\text{op}}$  is a duality.

We show that (i) implies (iii). Since  $\mathcal{A}$  is a dualizing  $k$ -variety,  $\text{DA}(X, -) \in \text{mod}_1 \mathcal{A}$ ,  $\text{DA}(-, X) \in \text{mod}_1 \mathcal{A}^{\text{op}}$  hold for any  $X \in \mathcal{A}$ . By Lemma 2.9, we have  $\text{mod } \mathcal{A} = \text{mod}_1 \mathcal{A}$  and  $\text{mod } \mathcal{A}^{\text{op}} = \text{mod}_1 \mathcal{A}^{\text{op}}$ . In general  $\text{GP}\mathcal{A} \subset \text{mod } \mathcal{A}$  holds. Let  $M \in \text{mod } \mathcal{A}$ . We show that  $M \in \text{GP}\mathcal{A}$ . Since  $\mathcal{A}$  is a dualizing  $k$ -variety,  $DM \in \text{mod } \mathcal{A}^{\text{op}}$  holds. By Lemma 2.15 (a),  $M^* \in \text{mod } \mathcal{A}^{\text{op}}$  holds. Thus by Lemma 2.15 (b),  $M \in \text{GP}\mathcal{A}$  holds.

We show that (iii) implies (ii). In general,  $\text{GP}\mathcal{A} \subset \text{mod } \mathcal{A} \subset \text{mod}_1 \mathcal{A}$  holds. Therefore by Lemma 2.9,  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  have weak kernels. Consider the functor  $D \circ (-)^* : \text{Mod } \mathcal{A} \rightarrow \text{Mod } \mathcal{A}$ . This functor induces an equivalence  $\text{proj } \mathcal{A} \xrightarrow{\sim} \text{proj } \mathcal{A}$ . In fact, if  $M \in \text{proj } \mathcal{A}$ , then  $M^* \in \text{proj } \mathcal{A}^{\text{op}}$ . By the assumption, we have  $D(M^*) \in \text{mod}_1 \mathcal{A} = \text{GP}\mathcal{A}$ . Since  $D : \text{Mod}_{\text{fg}} \mathcal{A}^{\text{op}} \rightarrow \text{Mod}_{\text{fg}} \mathcal{A}$  is a duality,  $D(M^*)$  is an injective object of  $\text{Mod}_{\text{fg}} \mathcal{A}$ . In particular,  $D(M^*)$  is a relative-injective object of  $\text{GP}\mathcal{A}$ . Since  $\text{GP}\mathcal{A}$  is Frobenius,  $D(M^*)$  is an object of  $\text{proj } \mathcal{A}$ . Thus we have a functor  $D \circ (-)^* : \text{proj } \mathcal{A} \rightarrow \text{proj } \mathcal{A}$ . This is an equivalence, since its quasi-inverse is given by  $(-)^* \circ D$ . Since  $\mathcal{A}$  is idempotent complete, the Yoneda embedding  $\mathcal{A} \rightarrow \text{proj } \mathcal{A}$ ,  $X \mapsto \mathcal{A}(-, X)$  is equivalence. Thus there exists an equivalence  $\mathbb{S} : \mathcal{A} \rightarrow \mathcal{A}$  such that the following diagram is commutative:

$$\begin{array}{ccc} \text{proj } \mathcal{A} & \xrightarrow{D \circ (-)^*} & \text{proj } \mathcal{A} \\ \simeq \uparrow & & \simeq \uparrow \\ \mathcal{A} & \xrightarrow{\mathbb{S}} & \mathcal{A}. \end{array}$$

For  $X, Y \in \mathcal{A}$ , we have the following isomorphisms which are functorial at  $X, Y$ :

$$\begin{aligned} \mathcal{A}(Y, \mathbb{S}X) &\simeq D(\mathcal{A}(-, X)^*)(Y) \\ &\simeq D(\text{Mod } \mathcal{A}(\mathcal{A}(-, X), \mathcal{A}(-, Y))) \\ &\simeq \text{DA}(X, Y). \end{aligned}$$

This means that  $\mathbb{S}$  is a Serre functor on  $\mathcal{A}$ .  $\square$

**2.4. Some observations on triangulated categories.** In this subsection, we state some propositions which we use later. We state one theorem for Frobenius categories. Let  $\mathcal{A}$  be an additive category and  $\mathcal{B}$  be a subcategory of  $\mathcal{A}$ . For two objects  $X, Y \in \mathcal{A}$ , we



denote by  $\mathcal{A}_{\mathcal{B}}(X, Y)$  the subspace of  $\mathcal{A}(X, Y)$  consisting of all morphisms which factor through an object of  $\mathcal{B}$ . We denote by  $\mathcal{A}/[\mathcal{B}]$  the category defined as follows: the objects of  $\mathcal{A}/[\mathcal{B}]$  are the same as  $\mathcal{A}$  and the morphism space is defined by

$$(\mathcal{A}/[\mathcal{B}])(X, Y) := \mathcal{A}(X, Y) / \mathcal{A}_{\mathcal{B}}(X, Y),$$

for  $X, Y \in \mathcal{A}$ .

Let  $\mathcal{F}$  be a Frobenius category,  $\mathcal{P}$  the full subcategory of  $\mathcal{F}$  consisting of the projective objects in  $\mathcal{F}$  and  $\underline{\mathcal{F}} := \mathcal{F}/[\mathcal{P}]$ . By Happel [H], it is known that  $\underline{\mathcal{F}}$  is a triangulated category. Assume that  $\mathcal{P}$  is idempotent complete. Let  $\mathbf{K}^b(\mathcal{P})$  be the homotopy category of complexes of  $\mathcal{P}$ . We denote by  $\mathbf{K}^{-,b}(\mathcal{P})$  the full subcategory of  $\mathbf{K}(\mathcal{P})$  consisting of complexes  $X = (X^i, d^i : X^i \rightarrow X^{i+1})$  satisfying the following conditions.

- (1) There exists  $n_X \in \mathbb{Z}$  such that  $X^i = 0$  for any  $i > n_X$ .
- (2) There exist  $m_X \in \mathbb{Z}$  and exact sequences  $0 \rightarrow Y^{i-1} \xrightarrow{a^{i-1}} X^i \xrightarrow{b^i} Y^i \rightarrow 0$  in  $\mathcal{F}$  for any  $i \leq m_X$  such that  $d^i = a^i b^i$  for any  $i < m_X$ .

We identify the category  $\mathcal{F}$  with the full subcategory of  $\mathbf{K}^{-,b}(\mathcal{P})$  consisting of  $X$  satisfying  $n_X \leq 0 \leq m_X$ . Then we have the following analogy of the well known equivalence due to [Bu, KV, R].

**Theorem 2.17.** [IY] *Let  $\mathcal{F}$  be a Frobenius category and  $\mathcal{P}$  the full subcategory of  $\mathcal{F}$  consisting of the projective objects. Assume that  $\mathcal{P}$  is idempotent complete. Then the composite  $\mathcal{F} \rightarrow \mathbf{K}^{-,b}(\mathcal{P}) \rightarrow \mathbf{K}^{-,b}(\mathcal{P})/\mathbf{K}^b(\mathcal{P})$  induces a triangle equivalence  $\underline{\mathcal{F}} \xrightarrow{\sim} \mathbf{K}^{-,b}(\mathcal{P})/\mathbf{K}^b(\mathcal{P})$ .*

Let  $\mathcal{U}$  be a triangulated category and  $\mathcal{X}$  be a full subcategory of  $\mathcal{U}$ . We call  $\mathcal{X}$  a *thick* subcategory of  $\mathcal{U}$  if  $\mathcal{X}$  is a triangulated subcategory of  $\mathcal{U}$  and closed under direct summands. We denote by  $\text{thick}_{\mathcal{U}} \mathcal{X}$  the smallest thick subcategory of  $\mathcal{U}$  which contains  $\mathcal{X}$ . Whenever if there is no danger of confusion, let  $\text{thick}_{\mathcal{U}} \mathcal{X} = \text{thick} \mathcal{X}$ .

**Lemma 2.18.** *Let  $\mathcal{T}, \mathcal{U}$  be triangulated categories and  $F : \mathcal{U} \rightarrow \mathcal{T}$  a triangle functor. Let  $\mathcal{X}$  be a full subcategory of  $\mathcal{U}$ . Then the following holds.*

- Assume that a map

$$F_{M,N[n]} : \mathcal{U}(M, N) \rightarrow \mathcal{T}(FM, FN[n])$$

*is an isomorphism for any  $M, N \in \mathcal{X}$  and any  $n \in \mathbb{Z}$ . Then  $F : \text{thick} \mathcal{X} \rightarrow \mathcal{T}$  is fully faithful.*

- If moreover  $\mathcal{U}$  is idempotent complete,  $\text{thick} \mathcal{X} = \mathcal{U}$  and  $\text{thick}(\text{Im}(F)) = \mathcal{T}$ , then  $F$  is an equivalence.

### 3. REPETITIVE CATEGORIES

**3.1. Repetitive categories.** We recall the definition of repetitive categories of additive categories. The aim of this subsection is to show Theorem 3.7.

**Definition 3.1.** Let  $\mathcal{A}$  be a  $k$ -linear additive category. The *repetitive category*  $\mathbf{RA}$  is the  $k$ -linear additive category generated by the following category: the class of objects is  $\{(X, i) \mid X \in \mathcal{A}, i \in \mathbb{Z}\}$  and the morphism space is given by

$$\mathbf{RA}((X, i), (Y, j)) = \begin{cases} \mathcal{A}(X, Y) & i = j, \\ \mathbf{DA}(Y, X) & j = i + 1, \\ 0 & \text{else.} \end{cases}$$

For  $f \in \mathcal{RA}((X, i), (Y, j))$  and  $g \in \mathcal{RA}((Y, j), (Z, k))$ , the composition is given by

$$g \circ f = \begin{cases} g \circ f & i = j = k, \\ (\mathcal{DA}(Z, f))(g) & i = j = k - 1, \\ (\mathcal{DA}(g, X))(f) & i + 1 = j = k, \\ 0 & \text{else.} \end{cases}$$

We describe fundamental properties of repetitive categories of Hom-finite categories.

**Lemma 3.2.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category. The following statements hold.*

- (a)  $\mathcal{RA}$  is Hom-finite.
- (b)  $\mathcal{RA}$  has a Serre functor  $\mathbb{S}$  which is defined by  $\mathbb{S}(X, i) := (X, i + 1)$ .
- (c) If  $\mathcal{A}$  is idempotent complete, then so is  $\mathcal{RA}$ .

*Proof.* (a) (b) These are clear by the definition.

(c) By the definition, an object of  $\mathcal{RA}$  is indecomposable if and only if it is isomorphic to an object  $(X, i)$ , where  $X$  is an indecomposable object of  $\mathcal{A}$  and  $i$  is some integer. Let  $X$  be an indecomposable object of  $\mathcal{A}$  and  $i$  be an integer. Since  $\mathcal{A}$  is idempotent complete and Proposition 2.10,  $\text{End}_{\mathcal{RA}}(X, i) = \text{End}_{\mathcal{A}}(X)$  is local. Therefore again by Proposition 2.10,  $\mathcal{RA}$  is idempotent complete.  $\square$

We see a relation between the categories  $\text{mod}\mathcal{A}$  and  $\text{mod}\mathcal{RA}$  and consequently, we show Theorem 3.7. Let  $\mathcal{A}$  be a  $k$ -linear additive category and  $i \in \mathbb{Z}$ . Put the following full subcategory of  $\mathcal{RA}$ :

$$\mathcal{A}_i := \text{add}\{(X, i) \in \mathcal{RA} \mid X \in \mathcal{A}\}.$$

An inclusion functor  $\mathcal{A}_i \rightarrow \mathcal{RA}$  induces an exact functor

$$\rho_i : \text{Mod}\mathcal{RA} \rightarrow \text{Mod}\mathcal{A}_i.$$

Since a functor  $\mathcal{A} \rightarrow \mathcal{A}_i$  defined by  $X \mapsto (X, i)$  is an equivalence, we denote an object  $(X, i)$  of  $\mathcal{A}_i$  by  $X$  for simplicity.

Since we have a full dense functor  $\mathcal{RA} \rightarrow \mathcal{A}_i$  given by  $(X, j) \mapsto X$  if  $j = i$  and  $(X, j) \mapsto 0$  if else, we have a fully faithful functor from  $\text{Mod}\mathcal{A}_i$  to  $\text{Mod}\mathcal{RA}$ . Therefore we identify  $\text{Mod}\mathcal{A}_i$  with the full subcategory of  $\text{Mod}\mathcal{RA}$  consisting of  $\mathcal{RA}$ -modules  $M$  such that  $M(X, j) = 0$  for any  $j \neq i$  and any  $X \in \mathcal{A}$ .

**Lemma 3.3.** *Let  $\mathcal{A}$  be an additive category and  $i, j \in \mathbb{Z}$ .*

- (a) *We have  $\rho_j|_{\text{Mod}\mathcal{A}_i} = \text{id}_{\text{Mod}\mathcal{A}_i}$  if  $j = i$  and  $\rho_j|_{\text{Mod}\mathcal{A}_i} = 0$  if else.*
- (b) *For any  $X \in \mathcal{A}$ , we have an exact sequence*

$$0 \rightarrow \mathcal{DA}_{i-1}(X, -) \xrightarrow{\beta} \mathcal{RA}(-, (X, i)) \xrightarrow{\alpha} \mathcal{A}_i(-, X) \rightarrow 0 \quad (3.1)$$

*in  $\text{Mod}\mathcal{RA}$ . In particular, we have  $\rho_j(P) \in \text{add}\{\mathcal{A}_j(-, X), \mathcal{DA}_j(X, -) \mid X \in \mathcal{A}\}$  for any  $P \in \text{proj}\mathcal{RA}$  and  $j \in \mathbb{Z}$ .*

- (c) *Each finitely generated  $\mathcal{A}_i$ -module is a finitely generated  $\mathcal{RA}$ -module.*

*Proof.* (a) The assertions follow from the definition of  $\rho_j$ .

(b) We construct morphisms  $\alpha, \beta$  in  $\text{Mod}\mathcal{RA}$ . For an object  $(Y, j)$  of  $\mathcal{RA}$ , define

$$\alpha_{(Y, j)} := \begin{cases} \text{id}_{\mathcal{A}(Y, X)} & j = i, \\ 0 & \text{else,} \end{cases} \quad \beta_{(Y, j)} := \begin{cases} \text{id}_{\mathcal{DA}(X, Y)} & j + 1 = i, \\ 0 & \text{else,} \end{cases}$$

and extend  $\alpha$  and  $\beta$  on  $\mathbf{RA}$  additively. We can show that  $\alpha$  and  $\beta$  are actually morphisms in  $\mathbf{ModRA}$ . By definitions of  $\alpha$  and  $\beta$ , for an object  $(Y, j)$  of  $\mathbf{RA}$ , we have the following exact sequence

$$0 \rightarrow D\mathcal{A}_{i-1}(X, (Y, j)) \xrightarrow{\beta_{(Y, j)}} \mathbf{RA}((Y, j), (X, i)) \xrightarrow{\alpha_{(Y, j)}} \mathcal{A}_i((Y, j), X) \rightarrow 0$$

in  $\mathbf{Modk}$ . Thus we have an exact sequence (3.1). Since  $\rho_j$  is exact, by applying  $\rho_j$  to the exact sequence (3.1) and by using (a), we have the assertion.

(c) This follows from (b).  $\square$

By the following lemma, we construct a filtration of a module over repetitive categories. For  $M \in \mathbf{ModRA}$ , put  $\text{Supp } M := \{i \in \mathbb{Z} \mid \rho_i(M) \neq 0\}$ .

**Lemma 3.4.** *Let  $M \in \mathbf{ModRA}$  and  $i \in \mathbb{Z}$ .*

(a) *If  $\rho_{i-1}(M) = 0$ , then there exists a short exact sequence*

$$0 \rightarrow \rho_i(M) \xrightarrow{\alpha} M \rightarrow N \rightarrow 0$$

*in  $\mathbf{ModRA}$  such that  $\rho_i(N) = 0$  and  $\rho_j(N) = \rho_j(M)$  for any  $j > i$ .*

(b) *Assume that  $\text{Supp } M$  is a finite set and put  $m := \max \text{Supp } M$  and  $n := \min \text{Supp } M$ . Then there exists a sequence of subobjects of  $M$ :*

$$0 = M_{n-1} \subset M_n \subset \cdots \subset M_{m-1} \subset M_m = M$$

*such that  $M_i/M_{i-1} \simeq \rho_i(M)$  for any  $i = n, n+1, \dots, m$ .*

*Proof.* (a) We construct a monomorphism  $\alpha : \rho_i(M) \rightarrow M$  in  $\mathbf{ModRA}$ . For an object  $(X, j)$  of  $\mathbf{RA}$ , define

$$\alpha_{(X, j)} := \begin{cases} \text{id}_{M(X, j)} & j = i, \\ 0 & \text{else,} \end{cases}$$

and extend this on  $\mathbf{RA}$  additively. Since  $\rho_{i-1}(M) = 0$ ,  $\alpha$  is a morphism of  $\mathbf{ModRA}$ . By the definition,  $\alpha$  is mono. Then we have an exact sequence  $0 \rightarrow \rho_i(M) \rightarrow M \rightarrow N \rightarrow 0$  in  $\mathbf{ModRA}$ , where  $N := \text{Cok}(\alpha)$ . By Lemma 3.3, we have  $\rho_j(\rho_i(M)) = \rho_i(M)$  if  $j = i$  and  $\rho_j(\rho_i(M)) = 0$  if else. Therefore by applying the functor  $\rho_j$  to this exact sequence, we have the assertion.

(b) This follows from (a).  $\square$

By the following two lemmas, we see that the functors  $\mathbf{ModA} \rightarrow \mathbf{ModRA}$  and  $\rho_i : \mathbf{ModRA} \rightarrow \mathbf{ModA}$  restrict to functors between  $\mathbf{modA}$  and  $\mathbf{modRA}$  under certain assumptions. For simplicity, we use the notation  $\mathbf{mod}_{-1}\mathcal{A} := \mathbf{ModA}$ ,  $\mathbf{mod}_{\infty}\mathcal{A} := \mathbf{modA}$  and  $\infty - 1 := \infty$ .

**Lemma 3.5.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category and  $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . Assume that  $D\mathcal{A}(X, -) \in \mathbf{mod}_{n-1}\mathcal{A}$  holds for any  $X \in \mathcal{A}$ . Then an inclusion functor  $\mathbf{ModA}_i \rightarrow \mathbf{ModRA}$  restricts to a functor  $\mathbf{mod}_n\mathcal{A}_i \rightarrow \mathbf{mod}_n\mathbf{RA}$  for any  $i \in \mathbb{Z}$ .*

*Proof.* Let  $n \in \mathbb{Z}_{\geq 0}$ . It is sufficient to show that  $\mathcal{A}_i(-, X) \in \mathbf{mod}_n\mathbf{RA}$  for any  $i \in \mathbb{Z}$ . In fact, any  $M \in \mathbf{mod}_n\mathcal{A}_i$  has an exact sequence  $P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$  with  $P_i \in \mathbf{proj } \mathcal{A}_i$  and hence  $M$  belongs to  $\mathbf{mod}_n\mathbf{RA}$  by Lemma 2.4 (a).

We show  $\mathbf{proj } \mathcal{A}_i \subset \mathbf{mod}_n\mathbf{RA}$  for any  $i \in \mathbb{Z}$  by an induction on  $n$ . If  $n = 0$ , then by Lemma 3.3 (c), we have the assertion. Let  $n > 0$ ,  $X \in \mathcal{A}$  and  $i \in \mathbb{Z}$ . By Lemma 3.3 (b), there exists an exact sequence

$$0 \rightarrow D\mathcal{A}_{i-1}(X, -) \rightarrow \mathbf{RA}(-, (X, i)) \rightarrow \mathcal{A}_i(-, X) \rightarrow 0.$$

By the inductive hypothesis,  $D\mathcal{A}_{i-1}(X, -) \in \text{mod}_{n-1}\mathcal{R}\mathcal{A}$  holds. Therefore we have  $\mathcal{A}_i(-, X) \in \text{mod}_n\mathcal{R}\mathcal{A}$  by Lemma 2.4 (c).

By an argument similar to the above, the assertion holds when  $n = \infty$ .  $\square$

**Lemma 3.6.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category,  $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . Assume that  $D\mathcal{A}(X, -) \in \text{mod}_n\mathcal{A}$  holds for any  $X \in \mathcal{A}$ . Then the functor  $\rho_i : \text{Mod}\mathcal{R}\mathcal{A} \rightarrow \text{Mod}\mathcal{A}_i$  restricts to a functor  $\text{mod}_n\mathcal{R}\mathcal{A} \rightarrow \text{mod}_n\mathcal{A}_i$  for any  $i \in \mathbb{Z}$ .*

*Proof.* Let  $n \in \mathbb{Z}_{\geq 0}$  and  $M \in \text{mod}_n\mathcal{R}\mathcal{A}$ . We have an exact sequence  $P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$  in  $\text{Mod}\mathcal{R}\mathcal{A}$ , where  $P_j \in \text{proj}\mathcal{R}\mathcal{A}$  for each  $j \geq 0$ . Since  $\rho_i$  is exact, we have an exact sequence  $\rho_i(P_n) \rightarrow \cdots \rightarrow \rho_i(P_1) \rightarrow \rho_i(P_0) \rightarrow \rho_i(M) \rightarrow 0$  in  $\text{Mod}\mathcal{A}_i$ . By the assumption and Lemma 3.3 (b),  $\rho_i(P_j) \in \text{mod}_n\mathcal{A}_i$  holds for any  $j \geq 0$ . Therefore  $\rho_i(M) \in \text{mod}_n\mathcal{A}_i$  holds by Lemma 2.4 (a).

By an argument similar to the above, the assertion holds when  $n = \infty$ .  $\square$

Note that in general  $\text{mod}\mathcal{R}\mathcal{A} = \text{mod}_1\mathcal{R}\mathcal{A}$  does not hold for a  $k$ -linear additive category  $\mathcal{A}$ . This is the case where  $\mathcal{A}$  is a dualizing  $k$ -variety by Theorem 3.7 below. Note that there exists an equivalence  $(\mathcal{R}\mathcal{A})^{\text{op}} \simeq \mathcal{R}(\mathcal{A}^{\text{op}})$  given by  $(X, i) \mapsto (X, -i)$ .

**Theorem 3.7.** *Let  $\mathcal{A}$  be a dualizing  $k$ -variety. Then the following statements hold.*

- (a)  $\mathcal{R}\mathcal{A}$  and  $(\mathcal{R}\mathcal{A})^{\text{op}}$  have weak kernels.
- (b)  $\mathcal{R}\mathcal{A}$  is a dualizing  $k$ -variety.

*Proof.* Note that since  $\mathcal{A}$  is a dualizing  $k$ -variety,  $D\mathcal{A}(-, X) \in \text{mod}_1\mathcal{A}$  holds for any  $X \in \mathcal{A}$  and  $\text{mod}_1\mathcal{A} = \text{mod}\mathcal{A}$  holds.

(a) Let  $X, Y \in \mathcal{R}\mathcal{A}$  and  $f : \mathcal{R}\mathcal{A}(-, X) \rightarrow \mathcal{R}\mathcal{A}(-, Y)$  be a morphism of  $\text{mod}\mathcal{R}\mathcal{A}$ . We show that  $K := \text{Ker}(f)$  is a finitely generated  $\mathcal{R}\mathcal{A}$ -module. For any  $i \in \mathbb{Z}$ , we have an exact sequence  $0 \rightarrow \rho_i(K) \rightarrow \rho_i(\mathcal{R}\mathcal{A}(-, X)) \rightarrow \rho_i(\mathcal{R}\mathcal{A}(-, Y))$  in  $\text{Mod}\mathcal{A}_i$ . By Lemma 3.6, we have  $\rho_i(\mathcal{R}\mathcal{A}(-, X)), \rho_i(\mathcal{R}\mathcal{A}(-, Y)) \in \text{mod}_i\mathcal{A}_i$ . Therefore  $\rho_i(K) \in \text{mod}_i\mathcal{A}_i$  for any  $i \in \mathbb{Z}$ , since  $\mathcal{A}_i \simeq \mathcal{A}$  is a dualizing  $k$ -variety. By Lemma 3.5,  $\rho_i(K) \in \text{mod}\mathcal{R}\mathcal{A}$  for any  $i \in \mathbb{Z}$ . Since  $K$  is a submodule of  $\mathcal{R}\mathcal{A}(-, X)$ ,  $\text{Supp } K$  is a finite set. Thus by Lemma 3.4 (b),  $K$  has a finite filtration by finitely presented  $\mathcal{R}\mathcal{A}$ -modules  $\{\rho_i(K) \mid i \in \mathbb{Z}\}$  and we have  $K \in \text{mod}\mathcal{R}\mathcal{A}$ . In particular,  $K$  is finitely generated and  $\mathcal{R}\mathcal{A}$  has weak kernels. Since  $(\mathcal{R}\mathcal{A})^{\text{op}} \simeq \mathcal{R}(\mathcal{A}^{\text{op}})$  holds and  $\mathcal{A}^{\text{op}}$  is a dualizing  $k$ -variety,  $(\mathcal{R}\mathcal{A})^{\text{op}}$  has weak kernels.

(b) By the definition of dualizing  $k$ -varieties,  $\mathcal{A}$  is Hom-finite and idempotent complete. By Lemma 3.2,  $\mathcal{R}\mathcal{A}$  is Hom-finite and idempotent complete with a Serre functor. Therefore by Proposition 2.16,  $\mathcal{R}\mathcal{A}$  is a dualizing  $k$ -variety.  $\square$

**3.2. Tilting subcategories.** The aim of this subsection is to show Theorem 3.10. Before stating the main theorem, we need the following definition.

Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category. We denote by

$$\rho : \text{Mod}\mathcal{R}\mathcal{A} \rightarrow \text{Mod}\mathcal{A}$$

the forgetful functor, that is,  $\rho(M) := \bigoplus_{i \in \mathbb{Z}} \rho_i(M)$  for any  $M \in \text{Mod}\mathcal{R}\mathcal{A}$ , where we regard an  $\mathcal{A}_i$ -module  $\rho_i(M)$  as an  $\mathcal{A}$ -module by the equivalence  $\text{Mod}\mathcal{A}_i \simeq \text{Mod}\mathcal{A}$ . Note that  $\rho$  is an exact functor. We denote by  $\text{GP}(\mathcal{R}\mathcal{A}, \mathcal{A})$  the full subcategory of  $\text{GP}(\mathcal{R}\mathcal{A})$  consisting of all objects  $M$  such that the projective dimension of  $\rho(M)$  over  $\mathcal{A}$  is finite, that is,

$$\text{GP}(\mathcal{R}\mathcal{A}, \mathcal{A}) := \{ M \in \text{GP}(\mathcal{R}\mathcal{A}) \mid \text{proj. dim}_{\mathcal{A}} \rho(M) < \infty \}.$$

We consider the following condition on  $\mathcal{A}$ :

- (G) : the projective dimension of  $D\mathcal{A}(X, -)$  over  $\mathcal{A}$  is finite for any  $X \in \mathcal{A}$ .

**Proposition 3.8.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category. Then  $\mathcal{A}$  satisfies (G) if and only if  $\text{proj } R\mathcal{A} \subset \text{GP}(R\mathcal{A}, \mathcal{A})$  holds. In this case, the following statements hold.*

- (a)  $\text{GP}(R\mathcal{A}, \mathcal{A})$  is a Frobenius category such that the projective objects is the objects of  $\text{proj } R\mathcal{A}$ .
- (b) The inclusion functor  $\text{GP}(R\mathcal{A}, \mathcal{A}) \rightarrow \text{GP}(R\mathcal{A})$  induces a fully faithful triangle functor  $\underline{\text{GP}}(R\mathcal{A}, \mathcal{A}) \rightarrow \underline{\text{GP}}(R\mathcal{A})$ .

*Proof.* The first assertion follows from Lemma 3.3 (b). Assume that  $\mathcal{A}$  satisfies (G).

(a) By the definition and since  $\rho$  is exact,  $\text{GP}(R\mathcal{A}, \mathcal{A})$  is extension closed subcategory of  $\text{Mod } R\mathcal{A}$  and has enough projectives and enough injectives. Clearly, an object of  $\text{proj } R\mathcal{A}$  is relative projective of  $\text{GP}(R\mathcal{A}, \mathcal{A})$ . Let  $Q$  be a relative projective object of  $\text{GP}(R\mathcal{A}, \mathcal{A})$ . There exists an exact sequence  $0 \rightarrow M \rightarrow P \rightarrow Q \rightarrow 0$  in  $\text{GP}(R\mathcal{A})$  with  $P \in \text{proj } R\mathcal{A}$ . We have  $M \in \text{GP}(R\mathcal{A}, \mathcal{A})$  and therefore this sequence splits. Consequently, the relative projective objects of  $\text{GP}(R\mathcal{A}, \mathcal{A})$  is the objects of  $\text{proj } R\mathcal{A}$ .

(b) This follows from (a).  $\square$

We regard  $\underline{\text{GP}}(R\mathcal{A}, \mathcal{A})$  as a thick subcategory of  $\underline{\text{GP}}(R\mathcal{A})$  by Proposition 3.8 (b) if  $\mathcal{A}$  satisfies (G). Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category. We consider the following condition on  $\mathcal{A}$ :

$$(\text{IFP}) : D\mathcal{A}(X, -) \in \text{mod } \mathcal{A} \text{ holds for any } X \in \mathcal{A}.$$

Note that if  $\mathcal{A}$  is a dualizing  $k$ -variety, then  $\mathcal{A}$  satisfies (IFP). We denote by  $\mathcal{M}$  the full subcategory of  $\text{Mod } R\mathcal{A}$  given by

$$\mathcal{M} := \text{add}\{ \mathcal{A}_0(-, X) \mid X \in \mathcal{A} \}.$$

We recall the definition of tilting subcategories of a triangulated category.

**Definition 3.9.** Let  $\mathcal{T}$  be a triangulated category. A full subcategory  $\mathcal{M}$  of  $\mathcal{T}$  is called a *tilting subcategory* of  $\mathcal{T}$  if  $\mathcal{T}(\mathcal{M}, \mathcal{M}[i]) = 0$  for any  $i \neq 0$  and  $\text{thick } \mathcal{M} = \mathcal{T}$ .

We establish the following result.

**Theorem 3.10.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category and assume that  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (IFP). Then the following holds.*

- (a) *If  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (G), then  $\mathcal{M} \subset \text{GP}(R\mathcal{A}, \mathcal{A})$  holds and  $\mathcal{M}$  gives a tilting subcategory of  $\underline{\text{GP}}(R\mathcal{A}, \mathcal{A})$ .*
- (b) *If each object of  $\text{mod } \mathcal{A}$  and  $\text{mod } \mathcal{A}^{\text{op}}$  has finite projective dimension, then  $\mathcal{M} \subset \text{GP}(R\mathcal{A})$  holds and  $\mathcal{M}$  gives a tilting subcategory of  $\underline{\text{GP}}(R\mathcal{A})$ .*

In the case where  $\mathcal{A}$  is a dualizing  $k$ -variety, we have the following corollary.

**Corollary 3.11.** *Let  $\mathcal{A}$  be a dualizing  $k$ -variety. If each object of  $\text{mod } \mathcal{A}$  and  $\text{mod } \mathcal{A}^{\text{op}}$  has finite projective dimension, then  $\mathcal{M}$  is a tilting subcategory of  $\underline{\text{mod}} R\mathcal{A}$ .*

Before starting the proof of Theorem 3.10, we prepare two lemmas. Let  $\mathcal{A}$  be a  $k$ -linear additive category and  $i \in \mathbb{Z}$ . Put the following full subcategories of  $R\mathcal{A}$ :

$$\mathcal{A}_{< i} := \bigvee_{j < i} \mathcal{A}_j, \quad \mathcal{A}_{\geq i} := \bigvee_{j \geq i} \mathcal{A}_j.$$

For  $M \in \text{Mod } R\mathcal{A}$  and  $i \in \mathbb{Z}$ , let  $\rho_{< i}(M) := \bigoplus_{j < i} \rho_j(M)$  and  $\rho_{\geq i}(M) := \bigoplus_{j \geq i} \rho_j(M)$ .

**Lemma 3.12.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category. Let  $M$  and  $N$  be finitely generated  $R\mathcal{A}$ -modules and  $i \in \mathbb{Z}$ . Assume that  $\rho_{\geq i}(M) = 0$  and  $\rho_{< i}(N) = 0$ .*

(a) *There exist epimorphisms*

$$\mathcal{R}\mathcal{A}(-, X) \rightarrow M, \quad \mathcal{R}\mathcal{A}(-, Y) \rightarrow N,$$

*for some  $X \in \mathcal{A}_{<i}$  and  $Y \in \mathcal{A}_{\geq i}$ .*

(b) *We have  $(\text{Mod}\mathcal{R}\mathcal{A})(M, N) = 0$  and  $(\text{Mod}\mathcal{R}\mathcal{A})(N, M) = 0$ .*

(c) *Assume  $M \in \text{mod}\mathcal{R}\mathcal{A}$ . Let*

$$\cdots \rightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0 \quad (3.2)$$

*be a minimal projective resolution of  $M$  in  $\text{mod}\mathcal{R}\mathcal{A}$ . Then we have  $\rho_{\geq i}(\text{Ker } f_l) = 0$  for  $l \geq 0$ . Moreover by applying a functor  $\rho_{i-1}$ , we have a minimal projective resolution of  $\rho_{i-1}(M)$  in  $\text{mod}\mathcal{A}_{i-1}$ .*

*Proof.* (a) Since  $M$  and  $N$  are finitely generated, there exist epimorphisms  $\mathcal{R}\mathcal{A}(-, X) \rightarrow M$  and  $\mathcal{R}\mathcal{A}(-, Y) \rightarrow N$ , where  $X$  and  $Y$  are in  $\mathcal{R}\mathcal{A}$ . Let  $W$  be an object of  $\mathcal{A}_{\geq i}$ . By Yoneda's lemma and the assumption, we have  $(\text{Mod}\mathcal{R}\mathcal{A})(\mathcal{R}\mathcal{A}(-, W), M) \simeq M(W) = 0$ . Therefore we can replace  $X$  with an object of  $\mathcal{A}_{<i}$ . Similarly, we can replace  $Y$  with an object of  $\mathcal{A}_{\geq i}$ .

(b) By (a), there exists an epimorphism  $\mathcal{R}\mathcal{A}(-, X) \rightarrow M$ , where  $X \in \mathcal{A}_{<i}$ . We have a monomorphism  $(\text{Mod}\mathcal{R}\mathcal{A})(M, N) \rightarrow (\text{Mod}\mathcal{R}\mathcal{A})(\mathcal{R}\mathcal{A}(-, X), N)$ . Since  $(\text{Mod}\mathcal{R}\mathcal{A})(\mathcal{R}\mathcal{A}(-, X), N) \simeq N(X) = 0$ ,  $(\text{Mod}\mathcal{R}\mathcal{A})(M, N) = 0$  holds. Similarly, by applying  $(\text{Mod}\mathcal{R}\mathcal{A})(-, M)$  to an epimorphism  $\mathcal{R}\mathcal{A}(-, Y) \rightarrow N$ , we have  $(\text{Mod}\mathcal{R}\mathcal{A})(N, M) = 0$ .

(c) By (a), there exists  $X_0 \in \mathcal{A}_{<i}$  such that  $P_0$  is a direct summands of  $\mathcal{R}\mathcal{A}(-, X_0)$ . We have  $\rho_{\geq i}(\mathcal{R}\mathcal{A}(-, X_0)) = 0$ . Therefore the submodule  $\text{Ker } f_0$  of  $\mathcal{R}\mathcal{A}(-, X_0)$  satisfies  $\rho_{\geq i}(\text{Ker } f_0) = 0$ . By using this argument inductively, we have that there exist  $X_l \in \mathcal{A}_{<i}$  such that  $P_l$  is a direct summands of  $\mathcal{R}\mathcal{A}(-, X_l)$  for any  $l \geq 0$ . Therefore we have  $\rho_{\geq i}(\text{Ker } f_l) = 0$  for  $l \geq 0$ .

For any  $l \geq 0$ , by Lemma 3.3,  $\rho_{i-1}(P_l)$  is a direct sum of  $\mathcal{A}_{i-1}(-, X)$  for some  $X \in \mathcal{A}$  and zero objects. Therefore each  $\rho_{i-1}(P_l)$  is a projective  $\mathcal{A}_{i-1}$ -module. Minimality comes from the minimality of the resolution (3.2).  $\square$

We see when  $\text{GP}(\mathcal{R}\mathcal{A})$  contains the representable functors on  $\mathcal{A}$ . Note that there exists an equivalence  $(\mathcal{R}\mathcal{A})^{\text{op}} \simeq \mathcal{R}(\mathcal{A}^{\text{op}})$  given by  $(X, i) \mapsto (X, -i)$ . Thus we have a duality

$$\text{Mod}_{\text{fg}}\mathcal{R}\mathcal{A} \xrightarrow{\text{D}} \text{Mod}_{\text{fg}}(\mathcal{R}\mathcal{A})^{\text{op}} \xrightarrow{\sim} \text{Mod}_{\text{fg}}\mathcal{R}(\mathcal{A}^{\text{op}}).$$

By this duality, a full subcategory  $\text{mod}\mathcal{A}_i$  of  $\text{mod}\mathcal{R}\mathcal{A}$  goes to a full subcategory  $\text{mod}(\mathcal{A}^{\text{op}})_{-i}$  of  $\text{mod}\mathcal{R}(\mathcal{A}^{\text{op}})$ .

**Lemma 3.13.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category.*

(a) *The following statements are equivalent.*

(i)  *$\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (IFP).*

(ii)  *$\mathcal{A}_i(-, X) \in \text{GP}(\mathcal{R}\mathcal{A})$  and  $\mathcal{A}_i(X, -) \in \text{GP}(\mathcal{R}\mathcal{A})^{\text{op}}$  hold for any  $X \in \mathcal{A}$  and  $i \in \mathbb{Z}$ .*

(iii)  *$\text{D}\mathcal{A}_i(X, -) \in \text{GP}(\mathcal{R}\mathcal{A})$  and  $\text{D}\mathcal{A}_i(-, X) \in \text{GP}(\mathcal{R}\mathcal{A})^{\text{op}}$  hold for any  $X \in \mathcal{A}$  and  $i \in \mathbb{Z}$ .*

(b) *If  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (IFP), then  $\rho_i(M) \in \text{GP}(\mathcal{R}\mathcal{A})$  holds for any  $M \in \text{GP}(\mathcal{R}\mathcal{A})$  and  $i \in \mathbb{Z}$ .*

*Proof.* Note that by Lemma 3.2,  $\mathcal{R}\mathcal{A}$  has a Serre functor  $\mathbb{S}$ . Thus by Lemma 2.15, we have an isomorphism of functors  $(-)^* \simeq \text{D}(- \circ \mathbb{S}^{-1}) : \text{Mod}_{\text{fg}}\mathcal{R}\mathcal{A} \rightarrow \text{Mod}_{\text{fg}}\mathcal{R}(\mathcal{A}^{\text{op}})$ . We have

$$(\mathcal{A}_i(-, X))^* \simeq \text{D}(\mathcal{A}^{\text{op}})_{-i-1}(X, -) = \text{D}\mathcal{A}_{-i-1}(-, X) \quad (3.3)$$

for any  $X \in \mathcal{A}$  and  $i \in \mathbb{Z}$ . Therefore (ii) and (iii) of (a) are equivalent.

(a) We show that (i) implies (ii). Let  $X \in \mathcal{A}$ . By Lemma 3.5,  $\mathcal{A}_i(-, X) \in \text{modR}\mathcal{A}$  holds. We have  $(\mathcal{A}_i(-, X))^* \in \text{mod}(\text{R}\mathcal{A})^{\text{op}}$ , by the equality (3.3) and Lemma 3.5. Therefore by Lemma 2.15 (b), we have  $\mathcal{A}_i(-, X) \in \text{GP}(\text{R}\mathcal{A})$ . Dually, we have  $\mathcal{A}_i(X, -) \in \text{GP}(\text{R}\mathcal{A})^{\text{op}}$ .

We show that (ii) implies (i). Let  $X \in \mathcal{A}$ . Take a minimal projective resolution of  $\mathcal{A}_i(-, X)$  in  $\text{modR}\mathcal{A}$ :

$$\cdots \rightarrow Q_2 \rightarrow Q_1 \xrightarrow{d_1} \text{R}\mathcal{A}(-, (X, i)) \rightarrow \mathcal{A}_i(-, X) \rightarrow 0.$$

By Lemma 3.3 (b), we have  $\text{Im } d_1 = \text{D}\mathcal{A}_{i-1}(X, -)$ . By Lemma 3.12 (c), applying  $\rho_{i-1}$ , we have  $\text{D}\mathcal{A}_{i-1}(X, -) \in \text{mod}\mathcal{A}_{i-1}$ . This means  $\text{D}\mathcal{A}(X, -) \in \text{mod}\mathcal{A}$ . Dually, we have  $\text{D}\mathcal{A}(-, X) \in \text{mod}\mathcal{A}^{\text{op}}$ .

(b) By Lemma 3.3 (b), we have  $\rho_i(P) \in \text{add}\{\mathcal{A}_i(-, X), \text{D}\mathcal{A}_i(X, -) \mid X \in \mathcal{A}\}$  for any  $P \in \text{proj R}\mathcal{A}$ . Therefore  $(\rho_i(P))^* \in \text{mod}(\mathcal{A}^{\text{op}})_{-i-1}$  holds by the equality (3.3) and the assumption. Let  $M \in \text{GP}(\text{R}\mathcal{A})$  and  $P_\bullet = (P_j, d_j : P_j \rightarrow P_{j+1})$  be a totally acyclic complex such that  $\text{Im } d_0 = M$ , where  $P_j \in \text{proj R}\mathcal{A}$ . By applying  $\rho_i$ , we have an exact sequence  $\rho_i(P_\bullet) = (\rho_i(P_j), \rho_i(d_j) : \rho_i(P_j) \rightarrow \rho_i(P_{j+1}))$  such that  $\text{Im } \rho_i(d_0) = \rho_i(M)$ . We have an exact sequence  $\cdots \rightarrow \rho_i(P_{-1}) \rightarrow \rho_i(P_0) \rightarrow \rho_i(M) \rightarrow 0$ . By Lemmas 2.4 (b) and 3.5,  $\rho_i(M) \in \text{modR}\mathcal{A}$  holds. By applying a functor  $(-)^*$  to  $0 \rightarrow \rho_i(M) \rightarrow \rho_i(P_1) \rightarrow \rho_i(P_2) \rightarrow \cdots$ , and using Lemma 2.4 (b) to the resulting exact sequence, we have  $(\rho_i(M))^* \in \text{mod}(\text{R}\mathcal{A})^{\text{op}}$ . Therefore we have  $\rho_i(M) \in \text{GP}(\text{R}\mathcal{A})$  by Lemma 2.15 (b).  $\square$

By Lemma 3.13, if  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (IFP), then  $\mathcal{M} \subset \text{GP}(\text{R}\mathcal{A})$  holds. We also denote by  $\mathcal{M}$  the subcategory of  $\underline{\text{GP}}(\text{R}\mathcal{A})$  consisting of objects  $\mathcal{A}_0(-, X)$  for any  $X \in \mathcal{A}$ . Then we show Theorem 3.10. We divide the proof into two propositions. Put  $\mathcal{T} := \underline{\text{GP}}(\text{R}\mathcal{A})$ .

**Proposition 3.14.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category and assume that  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (IFP). Then we have  $\mathcal{T}(\mathcal{M}, \mathcal{M}[i]) = 0$  for any  $i \neq 0$ .*

*Proof.* Let  $X \in \mathcal{A}$  and

$$\cdots \rightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} \mathcal{A}_0(-, X) \rightarrow 0$$

be a minimal projective resolution in  $\text{modR}\mathcal{A}$ . Put  $K^i := \text{Ker}(f^{i-1})$  for  $i \geq 1$ . By Lemmas 3.3 (b) and 3.12 (c), we have  $\rho_{\geq 0}(K^i) = 0$  for  $i \geq 1$ . Let  $Y \in \mathcal{A}$ . Since  $\rho_{< 0}(\mathcal{A}_0(-, Y)) = 0$  and Lemma 3.12 (b), we have

$$(\text{ModR}\mathcal{A})(K^i, \mathcal{A}_0(-, Y)) = 0, \quad (\text{ModR}\mathcal{A})(\mathcal{A}_0(-, Y), K^i) = 0,$$

for any  $i \geq 1$ . Therefore we have

$$\begin{aligned} \mathcal{T}(\mathcal{A}_0(-, Y), \mathcal{A}_0(-, X)[-i]) &= \mathcal{T}(\mathcal{A}_0(-, Y), K^i) = 0, \\ \mathcal{T}(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)[i]) &= \mathcal{T}(K^i, \mathcal{A}_0(-, Y)) = 0, \end{aligned}$$

for any  $i \geq 1$ .  $\square$

**Proposition 3.15.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category and assume that  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (IFP). If  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (G), then we have  $\text{thick}_{\mathcal{T}} \mathcal{M} = \underline{\text{GP}}(\text{R}\mathcal{A}, \mathcal{A})$ .*

*Proof.* Since  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (IFP), we have  $\mathcal{M} \subset \underline{\text{GP}}(\text{R}\mathcal{A}, \mathcal{A})$ . Therefore we have  $\text{thick } \mathcal{M} := \text{thick}_{\mathcal{T}} \mathcal{M} \subset \underline{\text{GP}}(\text{R}\mathcal{A}, \mathcal{A})$ .

Let  $i \in \mathbb{Z}$  and  $N \in \text{mod}\mathcal{A}_i$ . Assume that  $N$  has finite projective dimension over  $\mathcal{A}_i$ . Since the inclusion  $\text{mod}\mathcal{A}_i \rightarrow \text{modR}\mathcal{A}$  is exact, we have a resolution of  $N$  by objects of the form  $\mathcal{A}_i(-, X)$ , ( $X \in \mathcal{A}$ ) in  $\text{modR}\mathcal{A}$ . Therefore if  $N$  is an object of  $\text{GP}(\text{R}\mathcal{A}, \mathcal{A})$ , then  $N$  is in  $\text{thick } \mathcal{M}$  if  $\mathcal{A}_i(-, X)$  is in  $\text{thick } \mathcal{M}$  for any  $X \in \mathcal{A}$ .

Let  $M \in \mathbf{GP}(\mathbf{R}\mathcal{A}, \mathcal{A})$ . Since  $M$  is a factor module of a finitely generated projective  $\mathbf{R}\mathcal{A}$ -module,  $\mathrm{Supp} M$  is a finite set. Thus by Lemma 3.4 (b),  $M$  has a finite filtration by  $\rho_i(M)$  for  $i = n, n+1, \dots, m$ , where  $n = \min \mathrm{Supp} M$  and  $m = \max \mathrm{Supp} M$ . By Lemma 3.13 (b) and since  $\rho(M)$  has finite projective dimension over  $\mathcal{A}$ ,  $\rho_i(M) \in \mathbf{GP}(\mathbf{R}\mathcal{A}, \mathcal{A})$  for any  $i \in \mathbb{Z}$ . Therefore  $M$  is in  $\mathbf{thick} \mathcal{M}$  if  $\mathcal{A}_i(-, X)$  is in  $\mathbf{thick} \mathcal{M}$  for any  $X \in \mathcal{A}$  and  $i = n, n+1, \dots, m$ .

We show that  $\mathcal{A}_i(-, X)$  is in  $\mathbf{thick} \mathcal{M}$  for any  $X \in \mathcal{A}$  and  $i \in \mathbb{Z}$  by an induction on  $i$ . We first show  $\mathcal{A}_i(-, X) \in \mathbf{thick} \mathcal{M}$  for  $i \geq 0$ . Since  $\mathcal{A}_0(-, X) \in \mathcal{M}$ , we have  $\mathcal{A}_0(-, X) \in \mathbf{thick} \mathcal{M}$ . Assume that  $\mathcal{A}_j(-, X) \in \mathbf{thick} \mathcal{M}$  for  $0 \leq j \leq i-1$ . By Lemma 3.3, we have an exact sequence in  $\mathbf{GP}(\mathbf{R}\mathcal{A})$

$$0 \rightarrow \mathrm{D}\mathcal{A}_{i-1}(X, -) \rightarrow \mathbf{R}\mathcal{A}(-, (X, i)) \rightarrow \mathcal{A}_i(-, X) \rightarrow 0.$$

Since  $\mathrm{D}\mathcal{A}_{i-1}(X, -)$  has finite projective dimension over  $\mathcal{A}$  and by the inductive hypothesis, we have  $\mathrm{D}\mathcal{A}_{i-1}(X, -) \in \mathbf{thick} \mathcal{M}$ . Therefore  $\mathcal{A}_i(-, X)$  is in  $\mathbf{thick} \mathcal{M}$ .

Next we show that  $\mathcal{A}_{-i}(-, X) \in \mathbf{thick} \mathcal{M}$  for  $i > 0$ . Assume that  $\mathcal{A}_{-j}(-, X) \in \mathbf{thick} \mathcal{M}$  for  $0 \leq j \leq i-1$ . Let  $n$  be the projective dimension of  $\mathrm{D}\mathcal{A}_{-i}(-, X) \simeq \mathrm{D}(\mathcal{A}^{\mathrm{op}})_i(X, -)$  in  $\mathbf{mod}(\mathcal{A}^{\mathrm{op}})_i$  and

$$Q_n \xrightarrow{f} \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow \mathrm{D}\mathcal{A}_{-i}(-, X) \rightarrow 0$$

be a minimal projective resolution in  $\mathbf{mod}(\mathbf{R}\mathcal{A})^{\mathrm{op}} \simeq \mathbf{mod} \mathbf{R}(\mathcal{A}^{\mathrm{op}})$ . Put  $K := \mathrm{Ker} f$ . We have  $K \in \mathbf{GP}(\mathbf{R}(\mathcal{A}^{\mathrm{op}}))$  by Lemmas 2.7 (b) and 3.13 (a). By applying  $\rho$  to this resolution, we have  $K \in \mathbf{GP}(\mathbf{R}(\mathcal{A}^{\mathrm{op}}), \mathcal{A}^{\mathrm{op}})$ . Since the projective dimension of  $\mathrm{D}\mathcal{A}_{-i}(-, X)$  in  $\mathbf{mod}(\mathcal{A}^{\mathrm{op}})_i$  is  $n$  and by Lemma 3.12 (c), we have  $\rho_i(K) = 0$ . Moreover by Lemma 3.12 (c), we have  $\rho_{\geq i+1}(K) = 0$ . Therefore a  $\mathbf{R}\mathcal{A}$ -module  $\mathrm{D}K$  satisfies  $\rho_{< -i+1}(\mathrm{D}K) = 0$ . Since  $\mathrm{D}K$  is a finitely generated  $\mathbf{R}\mathcal{A}$ -module,  $\mathrm{Supp} \mathrm{D}K$  is finite. Thus by Lemma 3.4 (b),  $\mathrm{D}K$  has a finite filtration by  $\rho_j(\mathrm{D}K)$  for  $-i+1 \leq j \leq m$ , where  $m = \max \mathrm{Supp} \mathrm{D}K$ . By the inductive hypothesis,  $\mathrm{D}K \in \mathbf{thick} \mathcal{M}$  holds. We have an exact sequence in  $\mathbf{GP}(\mathbf{R}\mathcal{A})$

$$0 \rightarrow \mathcal{A}_{-i}(-, X) \rightarrow \mathrm{D}Q_0 \rightarrow \mathrm{D}Q_1 \rightarrow \cdots \rightarrow \mathrm{D}Q_n \rightarrow \mathrm{D}K \rightarrow 0,$$

where each  $\mathrm{D}Q_i$  is a projective  $\mathbf{R}\mathcal{A}$ -module. This means  $\mathcal{A}_{-i}(-, X) \simeq (\mathrm{D}K)[-n-1]$  in  $\mathbf{GP}(\mathbf{R}\mathcal{A}, \mathcal{A})$ . Therefore we have  $\mathcal{A}_{-i}(-, X) \in \mathbf{thick} \mathcal{M}$ .  $\square$

*Proof of Theorem 3.10.* (a) This follows from Propositions 3.14 and 3.15.

(b) Since each object of  $\mathbf{mod} \mathcal{A}$  has finite projective dimension,  $\mathbf{GP}(\mathbf{R}\mathcal{A}, \mathcal{A}) = \mathbf{GP}(\mathbf{R}\mathcal{A})$  holds. Thus the assertion follows from (a).  $\square$

*Proof of Corollary 3.11.* If  $\mathcal{A}$  is a dualizing  $k$ -variety, then  $\mathbf{GP}(\mathbf{R}\mathcal{A}) = \mathbf{mod} \mathbf{R}\mathcal{A}$  holds. The assertion directly follows from Theorem 3.10.  $\square$

**3.3. Happel's theorem for functor categories.** As an application of Theorem 3.10, we show Happel's theorem for functor categories. We need the following lemma.

**Lemma 3.16.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category and assume that  $\mathcal{A}$  and  $\mathcal{A}^{\mathrm{op}}$  satisfy (IFP). Let  $X, Y \in \mathcal{A}$ ,  $\mathcal{T} := \mathbf{GP}(\mathbf{R}\mathcal{A})$ . We have the following equality:*

$$\mathcal{T}(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)[n]) \simeq \begin{cases} \mathcal{A}(X, Y) & n = 0, \\ 0 & \text{else.} \end{cases}$$



*Proof.* By Proposition 3.14,  $\mathcal{T}(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)[n \neq 0]) = 0$  holds. Moreover we have

$$\begin{aligned} (\text{Mod } \mathcal{R}\mathcal{A})(\mathcal{A}_0(-, X), \mathcal{R}\mathcal{A}(-, (Y, 0))) &\simeq (\text{Mod}(\mathcal{R}\mathcal{A})^{\text{op}})(\text{DR}\mathcal{A}(-, (Y, 0)), \text{D}\mathcal{A}_0(-, X)) \\ &\simeq (\text{Mod}(\mathcal{R}\mathcal{A})^{\text{op}})(\mathcal{R}\mathcal{A}((Y, -1), -), \text{D}\mathcal{A}_0(-, X)) \\ &\simeq \text{D}\mathcal{A}_0((Y, -1), X) = 0, \end{aligned} \quad (3.4)$$

where we use Lemma 3.2 (b) and Yoneda's lemma. By Lemma 3.3 (b), if a morphism  $f : \mathcal{A}_0(-, X) \rightarrow \mathcal{A}_0(-, Y)$  in  $\text{Mod } \mathcal{R}\mathcal{A}$  factors through an object of  $\text{proj } \mathcal{R}\mathcal{A}$ , then  $f$  factors through  $\mathcal{R}\mathcal{A}(-, (Y, 0))$ . Thus by the equality (3.4), we have

$$\mathcal{T}(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)) = (\text{Mod } \mathcal{R}\mathcal{A})(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)).$$

By applying the functor  $(\text{Mod } \mathcal{R}\mathcal{A})(-, \mathcal{A}_0(-, Y))$  to the exact sequence of Lemma 3.3 (b), since  $(\text{Mod } \mathcal{R}\mathcal{A})(\text{D}\mathcal{A}_{-1}(X, -), \mathcal{A}_0(-, Y)) = 0$  holds, we have

$$\begin{aligned} (\text{Mod } \mathcal{R}\mathcal{A})(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)) &\simeq (\text{Mod } \mathcal{R}\mathcal{A})(\mathcal{R}\mathcal{A}(-, (X, 0)), \mathcal{A}_0(-, Y)) \\ &\simeq \mathcal{A}_0((X, 0), Y) \\ &\simeq \mathcal{A}(X, Y). \end{aligned}$$

□

We have the following result, which is a functor category version of Happel's theorem.

**Corollary 3.17.** *Let  $\mathcal{A}$  be a  $k$ -linear, Hom-finite additive category and assume that  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (IFP).*

(a) *If  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$  satisfy (G), then we have a triangle equivalence*

$$\mathbf{K}^b(\text{proj } \mathcal{A}) \simeq \underline{\text{GP}}(\mathcal{R}\mathcal{A}, \mathcal{A}).$$

(b) *If each object of  $\text{mod } \mathcal{A}$  and  $\text{mod } \mathcal{A}^{\text{op}}$  has finite projective dimension, then we have a triangle equivalence*

$$\mathbf{K}^b(\text{proj } \mathcal{A}) \simeq \underline{\text{GP}}(\mathcal{R}\mathcal{A}).$$

*Proof.* (a) Let  $\mathcal{F} := \underline{\text{GP}}(\mathcal{R}\mathcal{A}, \mathcal{A})$  and  $\mathcal{P} := \text{proj } \mathcal{R}\mathcal{A}$ . An inclusion functor  $\text{proj } \mathcal{A} \simeq \text{proj } \mathcal{A}_0 \rightarrow \mathcal{F}$  induces a triangle functor  $\mathbf{K}^b(\text{proj } \mathcal{A}) \rightarrow \mathbf{K}^{-,b}(\mathcal{P})$ . Then we have the following triangle functors

$$F : \mathbf{K}^b(\text{proj } \mathcal{A}) \rightarrow \mathbf{K}^{-,b}(\mathcal{P}) \rightarrow \mathbf{K}^{-,b}(\mathcal{P})/\mathbf{K}^b(\mathcal{P}) \rightarrow \underline{\mathcal{F}},$$

where the third is a quasi-inverse of Theorem 2.17. We denote by  $F$  the composite of these functors. We show that  $F$  is an equivalence by using Lemma 2.18.

Put  $\mathcal{U} := \mathbf{K}^b(\text{proj } \mathcal{A})$  and  $\mathcal{T} := \underline{\text{GP}}(\mathcal{R}\mathcal{A}, \mathcal{A}) = \underline{\mathcal{F}}$ . Note that  $\text{proj } \mathcal{A}$  is a subcategory of  $\mathcal{U}$ . We show that a map

$$F_{M,N[n]} : \mathcal{U}(M, N) \rightarrow \mathcal{T}(FM, FN[n])$$

is an isomorphism for any  $M, N \in \text{proj } \mathcal{A}$  and  $n \in \mathbb{Z}$ . By Theorem 2.17, a quasi-inverse of  $\mathbf{K}^{-,b}(\mathcal{P})/\mathbf{K}^b(\mathcal{P}) \rightarrow \underline{\mathcal{F}}$  is induced from the composite of the canonical functors  $\mathcal{F} \rightarrow \mathbf{K}^{-,b}(\mathcal{P}) \rightarrow \mathbf{K}^{-,b}(\mathcal{P})/\mathbf{K}^b(\mathcal{P})$ . Therefore we have  $F(\mathcal{A}(-, X)) = \mathcal{A}_0(-, X)$  for any  $X \in \mathcal{A}$ . For any  $X, Y \in \mathcal{A}$ , we have

$$\mathcal{U}(\mathcal{A}(-, X), \mathcal{A}(-, Y)) = \mathcal{A}(X, Y), \quad \mathcal{U}(\mathcal{A}(-, X), \mathcal{A}(-, Y)[n \neq 0]) = 0.$$

Consequently, by Lemma 3.16,  $F_{M,N[n]}$  is an isomorphism for any  $M, N \in \text{proj } \mathcal{A}$  and  $n \in \mathbb{Z}$ .

Since  $\text{proj } \mathcal{A}$  is Hom-finite and idempotent complete, so is  $K^b(\text{proj } \mathcal{A})$ . Clearly we have  $\text{thick}_{\mathcal{U}}(\text{proj } \mathcal{A}) = \mathcal{U}$ . Since  $\text{Im}(F|_{\text{proj } \mathcal{A}}) = \mathcal{M}$  holds, we have  $\text{thick}(\text{Im}(F)) = \mathcal{T}$  by Theorem 3.10 (a). Therefore  $F$  is an equivalence by Lemma 2.18.

(b) Since each object of  $\text{mod } \mathcal{A}$  has finite projective dimension, we have  $\underline{\text{GP}}(\text{R}\mathcal{A}, \mathcal{A}) \simeq \underline{\text{GP}}(\text{R}\mathcal{A})$ . Therefore we have the assertion by (a).  $\square$

**Corollary 3.18.** *Let  $\mathcal{A}$  be a dualizing  $k$ -variety. If each object of  $\text{mod } \mathcal{A}$  and  $\text{mod } \mathcal{A}^{\text{op}}$  has finite projective dimension, then we have the following triangle equivalence*

$$D^b(\text{mod } \mathcal{A}) \simeq \underline{\text{mod}} \text{R}\mathcal{A}.$$

*Proof.* If  $\mathcal{A}$  is a dualizing  $k$ -variety, then  $\text{GP}(\text{R}\mathcal{A}) = \text{mod } \text{R}\mathcal{A}$  holds. The assertion directly follows from Corollary 3.17.  $\square$

#### 4. PROOF OF THEOREM 1.1

Throughout this section, let  $k$  be an algebraically closed field. Let  $A$  be a finite dimensional hereditary  $k$ -algebra, that is,  $\text{gl.dim}(A) \leq 1$ . In this section, we apply Corollary 3.18 to  $\underline{\text{mod}} A$  and show Theorem 4.5.

We denote by  $\text{mod } A$  the category of the finitely generated  $A$ -modules and denote by  $\tau$  and  $\tau^{-1}$  the Auslander-Reiten translations on  $\text{mod } A$ . We call an indecomposable  $A$ -module  $M$  *preprojective* (resp. *preinjective*) if there exists an indecomposable projective  $A$ -module  $P$  (resp. injective  $A$ -module  $I$ ) and an integer  $i$  such that  $M \simeq \tau^i(P)$  (resp.  $M \simeq \tau^i(I)$ ). We call an indecomposable  $A$ -module  $M$  *regular* if  $\tau^i(M) \neq 0$  for any  $i \in \mathbb{Z}$ . Put the following subcategories of  $\text{mod } A$ :

$$\begin{aligned} \mathcal{P} &:= \text{add}\{M \in \text{mod } A \mid M \text{ is a preprojective module}\}, \\ \mathcal{I} &:= \text{add}\{M \in \text{mod } A \mid M \text{ is a preinjective module}\}, \\ \mathcal{R} &:= \text{add}\{M \in \text{mod } A \mid M \text{ is a regular module}\}. \end{aligned}$$

We denote by  $D^b(\text{mod } A)$  the bounded derived category of  $\text{mod } A$  and denote by  $\mathbb{S}$  a Serre functor of  $D^b(\text{mod } A)$ . We regard  $\text{mod } A$  as a full subcategory of  $D^b(\text{mod } A)$  by the canonical inclusion. Thus for any  $X \in D^b(\text{mod } A)$ ,  $X \in \text{mod } A$  if and only if  $H^i(X) = 0$  for any  $i \neq 0$ .

The following proposition is well known (see [ASS, Chapter VIII. 2.1. Proposition] [H, Chapter I, 5.2, Lemma]).

**Proposition 4.1.** *Let  $A$  be a representation infinite hereditary algebra. Then we have the following equalities.*

$$\begin{aligned} D^b(\text{mod } A) &= \bigvee_{i \in \mathbb{Z}} (\text{mod } A)[i], \\ \text{mod } A &= \mathcal{P} \vee \mathcal{R} \vee \mathcal{I}. \end{aligned}$$

We denote by  $\text{mod}_p A$  the full subcategory of  $\text{mod } A$  consisting of modules without non-zero projective direct summands. We define an additive functor

$$\Phi : \text{R}(\text{mod}_p A) \rightarrow D^b(\text{mod } A)$$

as follows. For  $X \in \text{mod}_p A$  and  $i \in \mathbb{Z}$ , let  $\Phi(X, i) := \mathbb{S}^i(X)$ . For  $X, Y \in \text{mod}_p A$  and  $i, j \in \mathbb{Z}$ , since  $\mathbb{S}$  is a Serre functor of  $D^b(\text{mod } A)$ , we have

$$\text{Hom}_{D^b(\text{mod } A)}(\mathbb{S}^i(X), \mathbb{S}^j(Y)) \simeq \begin{cases} \text{Hom}_{D^b(\text{mod } A)}(X, Y) & i = j, \\ D \text{Hom}_{D^b(\text{mod } A)}(Y, X) & j = i + 1, \\ 0 & \text{else,} \end{cases}$$

where the last isomorphism follows from Lemma 4.2. By using these isomorphisms, we define a map

$$\Phi_{(X,i),(Y,j)} : \text{Hom}_{\mathbf{R}(\text{mod}_p A)}((X,i), (Y,j)) \rightarrow \text{Hom}_{\mathbf{D}^b(\text{mod} A)}(\mathbb{S}^i(X), \mathbb{S}^j(Y)),$$

and we extend  $\Phi$  on  $\mathbf{R}(\text{mod}_p A)$  additively.  $\Phi$  is actually a functor, since a Serre duality is bifunctorial.

**Lemma 4.2.** *Let  $A$  be a representation infinite hereditary algebra. For any  $i < 0$  and  $j > 1$ , we have*

$$\mathbb{S}^i(\text{mod}_p A) \subset \text{add}(A) \vee \bigvee_{l < 0} \text{mod} A[l], \quad \mathbb{S}^j(\text{mod}_p A) \subset \text{add}(DA) \vee \bigvee_{l > 1} \text{mod} A[l].$$

*Proof.* The assertions come from Proposition 4.1.  $\square$

The first theorem of this section is the following. Put  $\mathbb{S}_1 := \mathbb{S} \circ [-1]$ . Note that  $H^0(\mathbb{S}_1(M)) \simeq \tau(M)$  and  $H^0(\mathbb{S}_1^{-1}(M)) \simeq \tau^{-1}(M)$  hold for any  $M \in \text{mod} A$ .

**Theorem 4.3.** *The functor  $\Phi : \mathbf{R}(\text{mod}_p A) \rightarrow \mathbf{D}^b(\text{mod} A)$  is an equivalence of additive categories.*

*Proof.* By the definition,  $\Phi$  is fully faithful. We show that  $\Phi$  is dense. Let  $X$  be an indecomposable object of  $\mathbf{D}^b(\text{mod} A)$ . By Proposition 4.1, there exist an indecomposable  $A$ -module  $M$  and an integer  $l$  such that  $X \simeq M[l]$ .

Assume that  $M$  is a preprojective module. There exist an indecomposable projective  $A$ -module  $P$  and  $i \geq 0$  such that  $M \simeq \mathbb{S}_1^{-i}(P)$ . If  $i+l > 0$ , then we have  $\mathbb{S}_1^{-(i+l)}(P) \in \text{mod}_p A$  and

$$\begin{aligned} \Phi(\mathbb{S}_1^{-(i+l)}(P), -l) &= \mathbb{S}^l(\mathbb{S}_1^{-(i+l)}(P)) \\ &= \mathbb{S}_1^{-i}(P)[l]. \end{aligned}$$

If  $i+l \leq 0$ , then we have  $\mathbb{S}_1^{-(i+l)}(\mathbb{S}(P)) \in \text{mod}_p A$  and

$$\begin{aligned} \Phi(\mathbb{S}_1^{-(i+l)}(\mathbb{S}(P)), -l+1) &= \mathbb{S}^{l-1}(\mathbb{S}_1^{-(i+l)}(\mathbb{S}(P))) \\ &= \mathbb{S}_1^{-i}(P)[l]. \end{aligned}$$

Next assume that  $M$  is a preinjective module. There exist an indecomposable injective  $A$ -module  $I$  and  $i \geq 0$  such that  $M \simeq \mathbb{S}_1^i(I)$ . If  $i-l \geq 0$ , then we have  $\mathbb{S}_1^{i-l}(I) \in \text{mod}_p A$  and

$$\begin{aligned} \Phi(\mathbb{S}_1^{i-l}(I), -l) &= \mathbb{S}^l(\mathbb{S}_1^{i-l}(I)) \\ &= \mathbb{S}_1^i(I)[l]. \end{aligned}$$

If  $i-l < 0$ , then we have  $\mathbb{S}_1^{i-l}(\mathbb{S}^{-1}(I)) \in \text{mod}_p A$  and

$$\begin{aligned} \Phi(\mathbb{S}_1^{i-l}(\mathbb{S}^{-1}(I)), -l-1) &= \mathbb{S}^{l+1}(\mathbb{S}_1^{i-l}(\mathbb{S}^{-1}(I))) \\ &= \mathbb{S}_1^i(I)[l]. \end{aligned}$$

Assume that  $M$  is a regular module. Then we have  $\mathbb{S}_1^{-l}(M) \in \mathcal{R} \subset \text{mod}_p A$  and  $\Phi(\mathbb{S}_1^{-l}(M), -l) = \mathbb{S}^l(\mathbb{S}_1^{-l}(M)) = M[l]$  holds. Therefore the functor  $\Phi : \mathbf{R}(\text{mod}_p A) \rightarrow \mathcal{D}$  is dense.  $\square$

Theorem 4.3 is an analog of the well known equivalence  $D^b(\mathcal{H}) \simeq \text{Rep } \mathcal{H}$  for a hereditary abelian category  $\mathcal{H}$  [Le, Theorem 3.1]. But they are quite different, since the definitions of  $\text{Rep } \mathcal{H}$  and  $R(\underline{\text{mod}} A)$  are quite different.

We recall the following proposition.

**Proposition 4.4.** [AR74, Propositions 6.2, 10.2] *Let  $\mathcal{A}$  be a dualizing  $k$ -variety and  $\mathcal{B} := \underline{\text{mod}} A$ . Let  $\mathcal{P}$  be the full subcategory of  $\mathcal{B}$  consisting of the projective modules. Then the following statements hold.*

- (a)  $\mathcal{B}/[\mathcal{P}]$  is a dualizing  $k$ -variety.
- (b) Assume that the global dimension of  $\underline{\text{mod}} A$  is at most  $n$ , then the global dimension of  $\underline{\text{mod}}(\mathcal{B}/[\mathcal{P}])$  is at most  $3n - 1$ .

Then we apply Corollary 3.18 to  $\underline{\text{mod}} A$ .

**Theorem 4.5.** *Let  $A$  be a representation infinite hereditary algebra. Then we have the following triangle equivalences*

$$\underline{\text{mod}} D^b(\underline{\text{mod}} A) \simeq \underline{\text{mod}} R(\underline{\text{mod}} A) \simeq D^b(\underline{\text{mod}}(\underline{\text{mod}} A)).$$

*Proof.* Since  $A$  is hereditary, a canonical functor  $\underline{\text{mod}}_p A \rightarrow \underline{\text{mod}} A$  induces an equivalence  $\underline{\text{mod}}_p A \simeq \underline{\text{mod}} A$ . Therefore the first equivalence comes from Theorem 4.3. By Proposition 4.4,  $\underline{\text{mod}} A$  is a dualizing  $k$ -variety such that the global dimension of  $\underline{\text{mod}}(\underline{\text{mod}} A)$  is at most two. Therefore we can apply Corollary 3.18 to the dualizing  $k$ -variety  $\underline{\text{mod}} A$ . We have the second equivalence.  $\square$

We say that two dualizing  $k$ -varieties  $\mathcal{A}$  and  $\mathcal{A}'$  are derived equivalent if the derived categories of  $\underline{\text{mod}} \mathcal{A}$  and  $\underline{\text{mod}} \mathcal{A}'$  are triangle equivalent.

**Corollary 4.6.** *Let  $A, A'$  be representation infinite hereditary algebras. If  $A$  and  $A'$  are derived equivalent, then  $\underline{\text{mod}} A$  and  $\underline{\text{mod}} A'$  are derived equivalent.*

**Remark 4.7.** If  $A$  is a representation finite hereditary algebra, then Theorems 4.3, 4.5 and Corollary 4.6 were shown by [IO].

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